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Studies in Education No. 12

THE TEACHING OF MATHEMATICS
IN
POST-PRIMARY SCHOOLS

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Studies in Education No. 12

THE TEACHING OF MATHEMATICS IN POST-PRIMARY SCHOOLS

By J. H. MURDOCH

NEW ZEALAND COUNCIL FOR
EDUCATIONAL RESEARCH

1950

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PREFACE

POST-PRIMARY SCHOOL TEACHING IN NEW ZEALAND IS BEING profoundly influenced by the new curriculum, itself the embodiment of new educational aims and ideas. This curriculum, though set forth as directions and suggestions from the Education Department, is of course very largely the result of recommendations made by teachers themselves. The 1936 Conference of the Secondary Schools' Association, in a unanimous resolution too lengthy and well-known to be quoted here, strongly urged a drastic remodelling of the traditional post-primary school course and anticipated the general form of the new regulations. It was a momentous pronouncement, making explicit and giving some definiteness to criticisms of the old type of education, and justifying hopes of the new.

We have now reached a stage where we can examine the new syllabus in the light of experience. The implications of our guiding principles and the complications resulting from their application are becoming more clear. There is in general little desire among teachers to return to the old ways, but there are inevitably doubts and some unhappiness. The more conservative members of the general public appear also to be somewhat critical both of the spirit and of the results of the new teaching.

In some subjects it has been relatively easy for teachers to modify their methods and aims as required. The prescription in English, for example, calls for more stress than formerly on what is described as communication English, and more attention to free reading, literature, and dramatic work. The change is mainly one of emphasis and can in practice be very slight. 'Social studies' has required of teachers a more difficult reorientation, but this appears to have been more or less satisfactorily effected despite some lingering doubts. The teaching of foreign languages, which are now taken by a smaller proportion of pupils, is being given a more liberal interpretation. Music, art, handwork, and physical education have acquired a new dignity and importance though we have not yet nearly enough trained specialist teachers to handle them. General science has been introduced as a core subject, and appears still to trouble many teachers.

This particular book deals with the remaining core subject. It seeks to explain the new conception of secondary school mathematics, and to point out the changes in form and method of presentation rendered necessary by the new outlook. It bears in mind that many teachers, in New Zealand at any rate, are called on to teach junior mathematics who make little claim to mathematical skill, and have no particular interest in the subject. In the narrow sense of the term it is not a piece of educational research, but rather an exposition of the principles underlying the 'new teaching' with particular reference to New Zealand conditions. The reader will not expect that ultimate aims or psychological doctrines will be justified in the following pages, though definite reference is made to both, nor will he regard any illustrative material as a blueprint of either matter or method. Controlling factors throughout have been the need to be brief and the desire to be clear, constructive, and helpful. References in the appendix are, in general, limited to books readily available in New Zealand, and within the compass of the ordinary teacher.

If apology is needed for some labouring of the obvious, I plead Sir Walter Scott's merciful exculpation—'for the man was human, and had been a schoolmaster'. After all, the obvious may be the most important and the most easily forgotten or overlooked. I am painfully conscious of grave limitations in my powers, experience, and qualifications, and would gladly have seen the work done by another. I am no mathematician, though I have taught mathematics for many years and have always had for it a sincere affection which with better teaching in my student days might have fostered some skill. But I am passably familiar with modern educational psychology and sociology, and my professional duties bring me into close contact with students and teachers of mathematics. With their help, and with the criticism of those competent mathematicians who read the whole work on behalf of the New Zealand Council for Educational Research, I may trust that I have written something of value to the teacher in the classroom. The general form of the work, and the evaluations offered, are an obvious reflection of my type of mind, and for all errors, whether of fact or of judgment, I accept responsibility.

The writer of such a book as this is beset by two contrasting temptations. He may limit himself to a factual statement of what is being done under existing conditions; or he may give rein to fancy, and, ignoring practical difficulties, sketch an ideal programme and procedure. But, in Browning's words,

> *The common problem, yours, mine, everyone's,*
> *Is not to fancy what were fair in life*
> *Provided it could be—but, finding first*
> *What may be, then find how to make it fair*
> *Up to our means—a very different thing.*

I have tried to face this problem squarely, to bear in mind classroom conditions while planning for better things. I know that classes may be too big or too heterogeneous, that the whole curriculum is overcrowded, that the time at the teacher's disposal is strictly limited, that children enter secondary schools variously prepared for post-primary work, that textbooks are unsatisfactory, examination prescriptions inconsistent, and examination papers too variable. I know that the modern school makes so many demands on pupils and teachers alike that elaborate lesson preparation is simply impossible, that interruptions and distractions and extra-classroom activities and changing programmes and staffing deficiencies hobble us throughout, and make the mere thought of further change distasteful. I know that the very provision of necessary extra accommodation may mean desperate months of intolerable hammering and clattering outside the classroom window. I even fear that these troubles may grow worse before they are cured. I know that the modern attitude towards work and discipline and play is itself a major problem, that parents and heads and inspectors and examiners have all to be placated, that teachers are not supermen but are entitled to the frailties common to humanity, that theorists who condemn without judgment and guide without wisdom call aloud for elimination—not necessarily painless. I know that mathematics is a difficult subject at the best of times, demanding a perseverance and a love of accuracy and thoroughness only painfully acquired by comfort-loving man, and that the whole trend of modern practice, whatever its theory, is increasingly unfavourable to it. I know, too, that far too many youngsters enter secondary schools quite unprepared for intellectual effort and unwilling to exert themselves, that many will drop all difficult subjects at the first opportunity and many will leave school too soon (so that it is unrealistic for the teacher to fret over failures), that the best and most modern of methods will not transform dullards or sluggards into scholars. This, and much more of the same tenor I know only too well—I have been a teacher all my life.

Practising teachers will, I hope, not be disappointed that more space has been given to general attitudes and principles than to the consideration of method. For this there are many good reasons. I am convinced that the urgent need at the moment is for a new attitude towards the teaching of mathematics among the rank and file of our teachers. Given this, we shall inevitably find a reformation in our teaching method. It is not so much teaching skill in the narrower sense that mathematicians commonly lack, as an insight into the psychology of normal adolescence and into the possibilities of mathematical teaching both as a preparation for the practical affairs of life and as an avenue to a fuller and more satisfying appreciation of life itself. Perhaps they lack most of all a consuming faith in the value of their own subject.

There are in fact many excellent publications (mainly English) dealing with specific teaching problems, and the teacher who really wants definite and detailed suggestions will find them there in abundance. The bibliography is offered as a guide.

Some critics, too, will be disturbed at my concern for the weaker pupils in school. I do not seek their welfare at the expense of the abler children or in a desire to pander to mediocrity. The new approach (advocated here and in most modern manuals) will meet the needs of the ablest pupils, but it must be intelligently and sympathetically interpreted.

As for the possibilities of doing anything with very weak pupils, I quote the testimony of a mathematics master who approves of my attitude and writes further: 'I have in mind a young teacher of one of our lowest ability domestic classes, who showed me an excellent set of examination papers that worried her because the average percentage might be too high—and a boy (IQ 79) with a 5 for arithmetic on his B.20 who tackled the algebra questions (in preference to some arithmetic ones) of the half-year exam. with simply amazing confidence and care. Yet teachers say such pupils are unteachable!'

I gladly acknowledge my indebtedness to the following, who assisted me with information, guidance, or criticism: Misses G. I. Warnock and G. L. Gardner and Messrs R. G. G. Dalgleish, L. M. McKillop, A. W. Short, R. D. Thompson, and W. H. Thomson—all mathematics teachers in Auckland—and Mr E. G. Hasell, King Edward Technical College, Dunedin; Messrs H. Henderson, L. le F. Ensor and M. D. Nairn, officers of the Education Department; Messrs R. A. Dickie, Principal of the Auckland Teachers' College, and F. R. Slevin, Headmaster of the Normal Intermediate School; the late Miss O. L. G. Adams, and Messrs H. C. Becroft, K. Hayr and R. P. Kania, my esteemed colleagues; Mr M. A. Bull, Rector of Timaru Boys' High School, and Drs J. T. Campbell of Victoria University College and E. R. Dalziel, late of Palmerston North Technical High School.

It is particularly difficult to express adequately my thanks to the permanent staff of the New Zealand Council for Educational Research for all they have done. Quietly, efficiently, and with unfailing patience and goodwill they have checked errors, pruned excrescences, and lightened a burden that would otherwise have fallen too heavily on a spare time worker. To Mr G. W. Parkyn, I am particularly indebted for a detailed criticism of my manuscript. I regret that some of the weaknesses he revealed were beyond my powers to cure, and some of my probably biased judgments too near to my heart to be safely removed.

J. H. MURDOCH

AUCKLAND, *March* 1950

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I

INTRODUCTORY

POPULAR OPINION CONCEDES SOME VALUE TO MATHEMATICS in two extreme instances. It is realized vaguely that modern science depends heavily on the mathematical expert and that industry demands what is known as 'applied mathematics'—not the pure variety that appeals to the philosophic reasoner. It is also very firmly held that ability to add and subtract, to calculate the proper change to be handed back, to estimate the cost of a particular job, and to secure the minimum assessment of income tax is worth while. In short, we have commonly a sharp distinction between simple practical skills that all must acquire and the abstruse calculations and hypothetical systems that should be left to the 'expert'. Anything in between is regarded as more or less valueless, except perhaps as a mental discipline—and there is a popular misconception even among teachers that modern psychology has exploded the whole doctrine of transfer of training.

This sharp distinction between the mathematical and the non-mathematical is paralleled in many secondary school subjects, and has disastrous consequences. Thus music enthusiasts today are struggling to change the popular opinion that the study and practice of music is the affair of a select and somewhat unmanly few, of no real significance, beyond that of an occasional pleasure, for workaday people who have a job to do. Yet we have entered an age where music, or what passes for music, is a constant background even in the hours of work or travel. Art likewise is regarded as the affair of a class apart, bohemian, untidy, and undisciplined. The study of foreign languages is similarly misunderstood, largely because in teaching we have stressed academic rather than spontaneous capacities, thereby rendering them intelligible only to a gifted few.

The advocate of each of these studies protests vigorously against this unreasoning attitude to his chosen subject, but may himself fall into the same error in his attitude to another subject. For this deplorable state of affairs the educational reformer lays much of the blame on the professional pedagogue, who appears to have a perfect genius for creating popular dislike of his subject. *Nihil tetigit quod non deformavit*, applied to many a specialist teacher, is perhaps too sweeping a condemnation, but it is not entirely without justification.

We cannot, then, and should not if we could, separate our secondary school pupils in the junior forms into linguists, mathematicians, scientists, artists, musicians, historians, geographers, engineers, farmers, athletes, and the like, and teach these narrowly and rigidly as distinct types. Such specialization belongs to a later date and concerns a gifted few; junior and middle school pupils must, if they are to become normal citizens, study fundamental subjects for which they may not be particularly adapted. This of course does not mean that we should not have pre-vocational courses specifically labelled 'Commercial', 'Home Science', 'Engineering', 'Academic', and the like. Vocational interests should in some degree be recognized, and there is value in calling an adolescent an engineer or a commercial student. But the common needs of adolescents must also be recognized, and this is the justification of the core prescription.

There is a strong element of defeatism in the average class teacher's handling of any but the brighter forms in a secondary school. For instance, during the past year assistants have assured me that the only method with a 'dull' form is to drive in class- and home-work and to insist on steady memorizing of dictated notes, that enlightened methods, presumed to be advocated in a training college, are too high-faluting to work under existing classroom conditions, that with certain forms it is not practicable to risk the loss of even a single period by allowing graduate students to try their hands at teaching, and finally, that 'these are dud forms and will never get anywhere in this subject'. One teacher writes to me: 'It is not so bad with the weakest pupils, as no one expects anything and is therefore pleased with very little.' The core syllabus alone, comprising some eight or nine highly diverse subjects, is manifestly too comprehensive for the weaker pupils, and forces the teacher to accept poor work or to camouflage failure by cramming. School organization normally demands, also, that 'general' teachers shall handle so many subjects that they cannot keep up with their work in all subjects—and they are sometimes themselves carrying the extra burden of attending university classes at the same time. They lack knowledge and require direction. Pardonably unsure of themselves, they fall back on uncritical acceptance and blind following of a textbook, or on the treatment that was common in their own schooldays. Senior masters, themselves too busy, cannot give much help and sometimes are temperamentally unfit to do so. Yet a bad start in a subject is fatal for all but the ablest pupils. And so apathetic and even obstructive classes may still be found, with teachers either conscientiously driving and plodding along, or less pardonably treating them as an opportunity to relax from the task of teaching brighter pupils. This is particularly noticeable with children remaining perforce at school till they reach the minimum leaving age.

That such attitudes can exist is an educational tragedy. The obvious remedies are either to drop the hopeless subject entirely or to handle it in a different and more effective way. The first method is simple, but if it is boldly applied some forms will soon have nothing left to study. Even games will be rejected as dull and over-tiring by many whose main purpose in life appears to be to avoid all serious effort, physical or mental. The second method is based upon the assumption that much of our failure is due to a faulty approach, and that we can fare better even with less time at our disposal, if we change our methods. If this belief is well founded there is still hope that a post-primary education for all is not a utopian dream. Certainly, if we really hope to raise the leaving age still higher we should first discover how to handle successfully our duller pupils up to fifteen years.

Our besetting sin as pedagogues, whatever the subject of instruction, is our persistent pettiness and lack of vision. We degrade wisdom to pedantry; we equate learning with painful and uninspired following in another's steps; we make teaching a blind driving of foolish sheep along a stony and endless road.[1] The debunking of idealism can become a disastrous practice.

The new core contains no foreign language and no specialized science, but still includes general mathematics. This implies a faith that mathematics, in a most elementary

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[1] cf. Professor H. E. Armstrong's letter to a Science Congress (quoted in J. H. Murdoch, *The High Schools of New Zealand*, 119. New Zealand Council for Educational Research, Wellington 1943). Also the lecture on mathematics (appendix) in J. W. A. Young's *The Teaching of Mathematics in the Elementary and the Secondary School*. London 1906, with supplement 1927.

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form at least, is both intelligible to almost all children and of practical value to them. The mathematics prescription, if less intensive than specialist teachers would desire, is for weak pupils quite formidable in extent. It is a serious attempt to meet essential needs of everyday people, and must be regarded as a specialized subject of first-rate importance. It presents a minimum standard of practical calculation.

Finally, at the back of all our trouble is the fact that the duties of teachers are now so multifarious. The three Rs no longer suffice. We cannot even limit our attention to the ever-increasing list of necessary subjects of study. The physical welfare of the child, the development of his emotional life, the analysis of his character and abilities, his vocational guidance and social adaptation demand more and more of our time and efforts. It may even be noted in passing that schools today suffer a minor invasion from outside specialists of all kinds—inspectors, liaison officers, important citizens, lecturers in this and that, student teachers, mental testers, vocational guiders, health authorities, social workers. There is an obvious danger that we shall be swamped by the multiplicity of our duties, and our pupils bewildered by the variety of their studies. It is not surprising that we should seek to jettison all that is not essential, and to integrate the rest. The new syllabus attempts both of these reactions in terms of producing the 'good citizen'.

As mathematicians, we must justify our subject in competition with others. We must associate it with the life and needs of the child, and get him to understand what it all means. And we must do this without loss of time. Can we do so? The Spens Report encourages us: 'We think that we have said sufficient to show how the amount of time now given to mathematics may be reduced without loss to the value of the subject. We realize that there are serious difficulties in carrying our proposals into effect, both in finding teachers who are willing to abandon the traditional methods in which they have been brought up, and in the requirements of existing examinations. But we are convinced that a reform in mathematical teaching is long overdue. We look to University Training Departments and Training Colleges to do their best to make reform possible by training a generation of men and women who will be equipped and ready to lead the way.'[2]

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[2] Gt. Britain, Board of Education, Report of the Consultative Committee on *Secondary Education with Special Reference to Grammar Schools and Technical High Schools*, 242. H.M.S.O., London 1939.

II

THE NEW BACKGROUND

WE SECONDARY SCHOOL TEACHERS MUST FIRST RECOGNIZE the new situation in which we labour, and this may be neither simple nor altogether encouraging. Secondary education for all sounds well as an ideal, so long as someone else has to deal with it in practice. The range of intelligence both in quality and in type creates in itself a tremendous problem for the schools. We have children ranging all the way, by almost imperceptible degrees, from the brilliant boy or girl who can win a university scholarship with ease to children who have difficulty in achieving even a very modest standard in the basic skills of the primary school. All these pupils are now our responsibility; we cannot afford to ignore any of them, or to sacrifice one group to the needs of another.

We must remember, too, the new status of arithmetic in the primary school. No longer does it share with English the leading role, while other subjects fill in the background or make 'noises off', and its treatment has also changed significantly. We cannot, therefore, reasonably expect secondary school entrants to have the same proficiency in the subject as did those of an earlier day, and complaints and denunciations are futile. The loss in arithmetical skill has presumably been offset by gain in other directions; whether the final balance favours new or old, and to what extent, is an educational problem of great complexity, not to be answered hastily in terms of personal bias or opinion.

Let us consider more closely the primary school syllabus as it stands today. An inspectorial directive gives the following time-allotment of school subjects in Forms I and II (my grouping):

46%	English	8 hours 35 minutes		total
	Social Studies	3 hours		11 hours 35 minutes
15%	Arithmetic			3 hours 40 minutes
8%	General science		boys	1 hour 55 minutes
[4%]			girls	55 minutes
	Drawing and Handwork	2 hrs. 30 min.		
15%	Music	1 hr. 15 min.	boys	3 hours 45 minutes
[19%]	Needlework—girls	1 hr. 0 min.	girls	4 hours 45 minutes
16%	Physical education	2 hrs. 0 min.		
	Intervals and milk	2 hrs. 5 min.		4 hours 5 minutes

The relatively short time given to arithmetic is at once obvious, amounting to an average of only 44 minutes per day. In the lower classes, even less time is allotted, so that assuming a 40-week year, the whole course, from Standard 1 to Form II, covers nominally $773\frac{1}{3}$ hours. Under the system of a couple of generations ago the time allotment varied, and it was notorious that arithmetic was kept up to standard by having time filched, where necessary, from other subjects —a practice that is today strictly forbidden. It amounted to perhaps 7 hours a week, or a total for the whole course of 1,680 hours—say twice the present allowance. Since that time the allocation has been steadily reduced.

It must also be remembered that the young primary school teacher of today typically lacks both the old-time accuracy and grasp of the elements and the keenness and drive that characterized the arithmetic teacher of former days. A few figures will give some indication of the present position.

In 1947, 20 per cent of Division A entrants to Auckland Training College had not studied mathematics beyond the core standard. Corresponding figures for 1948 and 1949 were 40 and 52 per cent respectively. Such students, it will be noted, have never faced an external examination in either arithmetic or mathematics. In May 1947, all Division A students of the College were given a test comprising the mental arithmetic paper in Exercise 86 of the Education Department's Form II textbook with the accompanying written paper B. The time allowed, 1 hour and 10 minutes in all, was regarded by the Methods Lecturer as reasonable for Form II children. The papers were marked on the right-wrong principle, no notice being taken of the methods used. Average scores were as follows:

	Mental	Written
Year I Women	44%	51%
,, II ,,	58	50
,, I Men	69	67
,, II ,,	65	69

For those who find comfort in comparative virtue, I quote the following results in a similar investigation in South Africa.[1] There, 691 students with an average IQ of 107-8 and aged 19·3 years, scored on an apparently similar test an average of 46·7%, with standard deviation 16·8 and quartile deviation 12·25. In the four mechanical operations—addition, subtraction, multiplication, division—the students averaged 50·6 as against the Standard V average of 73·4%.

In passing, it may be noted that the old Training College Entrance Examination, abolished in 1938, required a pass in nine listed subjects, of which two were arithmetic, and algebra and geometry. The prescription in those subjects was similar to that of the University Entrance Examination at that date. Exemption in a subject was allowed to 'any candidate who has been granted a higher leaving certificate or who has gained an equivalent qualification'.[2] For a partial pass, a candidate was required to pass in 'at least seven subjects, including not less than three of the subjects: English language and literature, arithmetic, geography, history and civics'. There can be no doubt that students who found the Form II arithmetic paper difficult could not face the papers set for the Entrance Examination; nor could core candidates face the algebra-geometry paper.

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[1] A. J. van Zyl, *Mathematics at the Crossroads*, Capetown 1942.

[2] 'Amended Regulations for the Training College Entrance Examination' in the *New Zealand Gazette*, No. 1, 8 January 1932, p. 2.

At least some secondary school teachers of mathematics are disturbed and frankly puzzled at the apparent official satisfaction with the position of their subject in teachers' training colleges. They point out that whereas even students who have qualified in the University Entrance Examination in any other of the core subjects are required to continue their studies at college, even the poorest of mathematicians are excused further study of mathematics, and even of arithmetic. They observe, too, that in all other subjects, specialist lecturers are in charge, whereas the general method lecturers are expected to handle method in arithmetic teaching as part only of their course. It is recognized that number work in the infant school receives special attention; but for middle and upper primary school teaching, students are ill prepared. Arithmetic appears to be regarded as a necessary evil, to be reduced to its absolute minimum; and teachers who have little conception of the nature and possibilities of mathematics are regarded as capable of teaching the elements. The same criterion of minimum essential qualification is applied in no other subject, and the attitude of students in training towards one of the great branches of human learning must be correspondingly worsened.

I leave the debate at this stage. Its bearing on the teaching of mathematics in secondary schools is important, but indirect. It may be suggested, however, that the almost complete neglect of mathematics at the training college and the high percentage of students who have not studied it beyond the core prescription together emphasize the importance and indeed necessity of treating core mathematics seriously in post-primary schools. It may interest readers to note that entrants to the Auckland Teachers' Training College in 1949 had university qualifications as follows:

Women: 30 units on entrance, sat for 56 units in 1949; no mathematics units on entrance, sat no mathematics.

Men: 68 units on entrance, 7 in mathematics—held by three students;
sat for 83 units in 1949, including 1 unit in mathematics, in which the candidate failed.

Now let us consider the programme of work set out for Form II in the Education Department's revised syllabus of 1948. It calls for treatment of:

(1) Numeration and notation to millions;

(2) Practice in mechanical operations; (accuracy and speed in the four rules—simple, and applied to money, with small numbers; and use of ready reckoners.)

(3) Very simple fractions such as $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}$; $\frac{1}{2}-\frac{1}{4}-\frac{1}{8}$; similar handling of decimals (to three places) such as $\frac{\frac{2}{3}}{\frac{7}{9}}=?$; mental decimalization of money; expression of percentages of the simplest types: $\frac{1}{8}=0{\cdot}125=12\frac{1}{2}\%$;

(4) Ratio and simple direct proportion, again of the simplest;

(5) 'Social arithmetic', essentially elementary dealing with simple weights and volumes and simple monetary transactions—the aim being 'to familiarize pupils with ideas and

processes that will enter into their daily lives rather than to give practice in computation';

(6) Spatial knowledge, covering simple areas, scale drawing, volume and cubic measure;

(7) Familiarity with the most elementary terms associated with examples of the sections above.

Note the insistence throughout on *simplicity* of mechanical operations. Quite against the real intentions of the Syllabus Revision Committee, this placing of emphasis has led in some quarters to an almost morbid fear of mechanical drill. On the other hand, it is claimed in some places that the standards of speed and accuracy in computation are rising. The emphasis on simplicity, however, does not extend to the treatment of social and economic problems. (The Auckland Training College graduate students were badly floored by a series of questions covering information given in this section. *Par charité*, I give no statistics.)

The real significance of this scheme is illustrated in the school Arithmetic Book for Form II, published in 1947. The double page preceding Exercise I shows 'A new pylon for the 66,000 volt line from Lake Coleridge power-station to the West Coast'; and it adds, 'Power lines feed cities, towns, and country districts with electricity. The amount used in each house or factory is recorded on a Meter (see page 146)'. Facing this appears the following:

'Here are some of the questions that this book will help you to answer: What is the weight of a gallon of water?
What is the product of 4·23 and 4·5?
How do you open a savings-bank account?
What is needed to find the circumference of a circle?
How is an angle measured?

How are accounts of a school club kept?
What is the National Budget?
What is decimal coinage?
How is commission calculated?
What is the international date line?
How do you read a gas meter and an electric-light meter?'

This linking of arithmetic with everyday life is an essential feature of the book and marks a distinct break with the traditional approach. Numerous examples of mechanical exercises in fundamental operations are provided including very simple fractions and decimals, and frequent revision exercises indicate the work of an experienced classroom teacher. But long mechanical operations without reference to life or to probable mathematical situations are gone: the exercises deal with shopping transactions, wages and rent, sport and travel. Yet a certain unreality still remains. Of the examples on pages 39, 40, 41, 42, 44 very few deal with real life situations. Questions such as the following still remain: If 2½ pints of vinegar cost 1s. 10½d., find the price per pint. Explanations of applied arithmetic are illustrated, and given unhurriedly. For example, Exercise 60, headed 'Time', explains (with a map) local time, world time, and the 24-hour clock system. It is followed by lessons on 'Reading the Gas Meter, Electric Light and Power'. 'Keeping Accounts' (lesson 65) explains both household and national budgets (6½ pages), and is followed by a lesson on 'Rates and Taxes' (4 pages), in which the three main systems of local rating are explained, and the information on the back of a rate demand is reproduced.

A few general considerations may now be noted. First, there is always a danger that reform will go too far, throwing away the virtues along with the vices of the old methods. In particular, the new live approach may easily lead to an actual neglect of the fundamental drill without which accuracy and reasonable speed are lost.

Incidentally, it should be noted that the criticism of purely mechanical drill can readily go too far. The alleged drudgery may exist for the teacher rather than for the child. Certainly repetition and mechanical accuracy have their own appeal for the young child, who enjoys above all else the feeling of complete mastery.[3] That feeling may be denied in the new primary school, and the seeds sown of that sense of helplessness and frustration found later among so many children. Repetitive work can be quite thrilling, and is essential to success. Everyone who has not been misled by a superficial study of educational theory knows this quite well. In games we still insist that practice alone brings perfection. The *Handbook of Suggestions* stresses the need of mechanical skill, with due reservations. If the child does not master tables 'at the right time, he is hampered in all subsequent work and wastes, in the course of school life, far more time than their accurate memorization would have required',[4] and the same may be held of mechanical rules.

Associated with this is the obvious second danger, that too much of the arithmetic time will be devoted to efforts to explain such topics as those noted above. Teachers nursed on the social studies and weak in arithmetic will be tempted to stress the social element in the textbook at the expense of the arithmetic proper. But after all, the function of the arithmetic teacher is surely to teach arithmetic; and indeed, the very introduction of applied arithmetic in the syllabus makes the effective teaching of the fundamentals absolutely essential. The simplicity of the pure arithmetic contrasts strongly with the detailed social knowledge offered in the book. Let me illustrate this matter with a specific example, for it is of considerable importance. A book that appeals to me very strongly as an introduction to mathematics is J. P. McCormack's *Mathematics for Modern Life*.[5] This book stresses and illustrates from its very first page the all-pervasiveness of mathematics in modern life. Varied examples are taken from home life, travel, architecture, engineering, manufacturing, social and economic life, nature, science, aviation, business, cooking, and so on. Dealing with algebraic numbers, it introduces a reference (page 126) to music and gives a rule for the memorizing of the keys: 'Every time you raise the key a whole tone, add two sharps.' This exercise in the book is distinctly branded 'Optional'. Now, for children familiar with musical notation and practice this is an excellent example of the use of algebraic formula, but if it were forced by a foolish teacher upon children who knew nothing of music, the result could only be distress and confusion of the pupils. They would improve neither their music nor their arithmetic.

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[3] Teachers interested in this problem are referred to T. P. Nunn's *Education: Its Data and First Principles*, Third Edition, London 1945,—especially Chapter VI, 'Routine and Ritual'.

[4] Gt. Britain, Board of Education, *Handbook of Suggestions for the Consideration of Teachers and Others Concerned in the Work of Public Elementary Schools*, 506. H.M.S.O., London 1937.

The mathematical treatment of a familiar operation will add clarity and precision to the child's conception, and the application in itself makes for a surer grasp of the abstract method. But the example should be familiar, or we risk the logical error of *obscurum per obscurius*. A grasp of the principles is more important than a knowledge of types of application, and the appeal to the latter is justifiable mainly because it may strengthen the former. In technical language, there is a danger that a particular concept is so closely bound to a fundament that its general application is obscured. For complete learning, we must pass from the specific examples to the underlying principle, and we must have the insight necessary to reverse the process by applying general principles to new material, i.e., new 'fundaments'.

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[5] J. P. McCormack, *Mathematics for Modern Life*, New York 1937.

A third matter of some importance is the nature of the practical applications selected. These are predominantly of the social studies type, covering household problems and personal finance. Much of this is strictly outside the immediate interest or experience of Form II pupils, and might be withheld till a later date. More serious, the general tendency of this emphasis may be to establish false standards in the minds of pupils. The most dangerous of all propaganda is that which lies unsuspected in implied attitudes. We may readily overstress economic and commercial considerations with children. Citizenship means—even for the poorest intellectually—much more than shopping wisely, being handy about the house, and knowing how to minimize taxation payable. (The legal adviser of the newspaper *The Standard* on 27 February 1947 affirmed a normal and ethical struggle between the taxpayer, anxious to avoid taxation, and the Commissioner of Taxes, anxious to gather a full harvest. So much for our concepts of right and duty!)

Finally, the introduction of a section on 'spatial knowledge' is not without its dangers. This is in line with the oversea educational movement to amplify the primary school syllabus in arithmetic with some elementary geometrical concepts. A previous attempt to bring such work into the syllabus under the heading of geometry failed lamentably. Ignorant of the principles involved, teachers furbished up their elementary textbook knowledge and finished with a set of constructions, rules and definitions—the barren result of the uninformed transplantation of ideas. The treatment might well descend to the culling of a few simple manipulative examples from a textbook.

On the other hand, the new textbook has many virtues, of which the following may be listed: it does seek to make the subject meaningful and to illustrate its practical applications; it provides training in some useful knowledge; it makes use of such practical and everyday devices as ready-reckoners; it introduces graphical work, which can be so important in everyday life, and which leads to post-primary mathematics; it gives the pupil a reminder of the importance of approximations, so little understood generally; it avoids the error of meaningless and even fantastic mechanical operations, at the expense, perhaps, of an equally fantastic and unreal simplicity. Its aim is to represent arithmetic as a means to an end and to encourage pupils to check their results at the bar of common sense. The spirit underlying the course is most commendable. At the same time, lest we suffer from our virtues, I would remind educationists that children (and quite dull ones at that) can enjoy mathematical puzzles, the fun of playing with numbers, and the irrational appeal of the fantastic. Some of our reformers are too grimly set in their hatred of the non-practical.

The textbook has still to pass the test of classroom application, and at the present time a Departmental Revision Committee is critically examining the books for Forms I and II. The latter unfortunately contains lapses in accuracy of statement, advocates some unsatisfactory methods, omits some desirable if not essential work, and in the second part usurps the function of the teacher. In primary and intermediate school arithmetic, much of the work is reducible to mechanical operations and standardized methods, and should be so reduced. Complete mastery of the four fundamental processes is obviously essential, and any failure here is not to be pardoned even on the plea of the new teaching. Likewise, children must learn the common weights and measures and be able to convert them. They must also master the manipulation of fractions and decimals. Such skills are essential in everyday life, may be secured at this stage, and can become a permanent possession. I do not suggest that this mastery is always easily obtained. It calls for great patience, steady and conscientious drilling, and pedagogic skill.

Standard, straightforward, universal methods are essential. Fancy or specialized methods are out of place: they burden the memory, complicate learning, and obscure the generality of arithmetical procedures. For example, why formulate special rules for the decimalization of money? Decimalization in itself is a simple process, and can be applied in the same way to all fractions, moneys, weights and measures as required. In ordinary life, many of these specific applications will never be required of the average citizen. Where fundamental skills are mastered, transfer is readily secured. But there must be real mastery.

Very considerable agreement has now been reached among teachers and educational psychologists as to the methods that are most satisfactory. Where differences are permissible, the relative advantages and disadvantages of alternative methods can be pointed out so that teachers may choose with judgment. Thus a report of the General Teaching Committee of the British Mathematical Association recognizes and illustrates in addition to the use of factors, three methods of long multiplication of money—(1) the ten-ten method, which is simple, 'but with a multiplier running into thousands the preliminary forming of the rows is lengthy, and any mistake in the forming of the rows leads to an answer completely wrong'; (2) the 'wholesale or top-line' method, where the sum is worked directly. 'Preliminary drill is needed in the wide-spacing and the use of the figures in the top line as multipliers', but 'on the other hand each monetary unit is disposed of in turn, which is a definite convenience'; (3) the method of 'practice' which is 'elegant with a conveniently divisible sum of money', is almost essential in some instances, e.g. cost of 3 tons 15 cwt. 1 qr. at £4 6s. 5d. a ton, and is a method that may be applied very widely. On the other hand, however, 'it leaves something to the individual intelligence and it is not therefore easily made mechanical'. The Committee recommend 'that either the "Ten-ten" or the "Wholesale" method should be learned at the "4 Rules" stage: and feel that teachers should test for themselves the latter. The "Practice" method must certainly be taught, but at a later stage.'[6] The wholesale method is stated to be more recent but to be finding increased favour.

It is obviously important that really sound methods should be selected and thoroughly taught, and highly desirable that, at any rate in the same school, uniform methods should be in use in all classes. This may involve some slight limitation of the freedom of individual teachers, but only in matters of insignificant detail, where the benefits of conformity quite outweigh the claims of individualism. We readily obey traffic regulations in the common interest, with no feeling of repressed personality. Organization is not regimentation. Even where there is room for differences of opinion among experienced and capable teachers, an agreed uniformity may be desirable.[7]

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[6] Mathematical Association, report on *The Teaching of Arithmetic in Schools*, 22. London 1932.

Such agreement is usually sought in individual schools, where staff conferences may decide. Sometimes a wider conformity is sought, covering all schools contributing to a particular secondary school. I suggest that here is a legitimate field for more positive and definite guidance by the Department. Several moves are possible. It might issue a supplementary pamphlet setting out approved methods as a help to those who need guidance. A committee comprising representatives of primary, intermediate, and post-primary schools could handle the task of selection. Again, it might publicize significant developments of methodology. For instance *The Teaching of Arithmetic in Schools* might well be brought to the notice of teachers. Finally, it might appoint competent mathematics lecturers to the five training colleges and leave these to train future teachers in sound methods. Whatever is done, help and guidance for the average primary school teacher is essential, and co-ordination between primary and secondary schools must be strengthened. At present, secondary schools receive entrants who differ greatly not only in their mastery of the elements but also in methods of working. The Headmaster of an Auckland grammar school reports that in his preliminary classification test pupils' marks range from 0 to 100 per cent. The Revision Committee already referred to has found that there are in use in primary schools 24 methods for addition, 31 for subtraction, 9 for division and 14 for multiplication. The mind reels at such figures. It is hard to believe that even

[7] cf. *Handbook of Suggestions*, 507, 'methods should be standardized'.

pedagogic perversity can discover and practise so many variations.

As a temporary guide to teachers who are undecided among competing methods, I would suggest the following general criteria. (More specific notes on Form II methods will be found in the Appendix.)

★A method should be simple and clear, easily understood by the children and within the range of their acquired skill.

★It should be of general application, at least to all such examples as will be met in their school careers.

★It should be definite in the early stages, i.e. beginners should not be required to decide, on the actual figures before them, which method to use.

★It should be consistent. Thus, if complementary subtraction is to be taught, it should be used wherever subtraction occurs, and mental arithmetic should encourage the method. This involves care in the phrasing of questions.

★It should, further, be soundly based. *Ad hoc* explanations which may encourage misunderstandings of the real process should be avoided. In the early teaching of fractions this error is readily made.[8]

★The method should look forward to what will be required at the secondary school level. Primary schooling today is preparatory and not final, and we dare not keep our educational stages rigidly self-centred. Later teaching may legitimately though unhappily call for a modification of primary school methods, but should not involve their total rejection. The Mathematical Association's report says: 'No attempt should be made to alter those habits which involve a definite process of thought, as opposed to those which concern merely mechanical figure work. The method of subtraction is an example of the former; provided a boy can subtract correctly, it does not matter whether he thinks of it as taking away or as complementary addition.'[9]

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[8] See Mathematical Association, op. cit., 23-4.

*The method should be neat and logical. In particular, careless use of the sign = must be avoided; and, of course, so also must careless figuring and poor reducing signs.

*Finally, it should be designed to minimize the risk of error and the amount of mechanical working. Arithmetic has enough difficulties of its own without adding to them the dangers of complicated methods.

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[9] ibid., 13.

III

THE POST-PRIMARY CURRICULUM

§1 CORE MATHEMATICS

UNDER THE NEW REGULATIONS, ISSUED BY THE EDUCATION Department in 1945, the compulsory core mathematics course becomes educationally of the greatest importance.[1] It represents the limit of mathematical training for a large and increasing percentage of the post-primary school population in New Zealand. Though designed explicitly for 'those who may not have much mathematical ability'—the wording is kindly—it has become in fact the whole course for many who have the ability to reach the School Certificate standard in this subject or go even higher.

The curricula of our schools are still in process of modification, so that the final consequences of the new regulations can only be inferred. It would seem that the standard set in School Certificate mathematics, and its relative difficulty will be an important factor. We may assume that this examination will become, for all but the strongest and the weakest, the normal goal of secondary school entrants. Those who are marking time till they reach their fifteenth year will not be troubled by this standard, and University Entrance and Scholarship candidates will for very different reasons be little concerned; but the majority of intermediate students will have their courses greatly influenced by the School Certificate requirements. (This applies least to short-course pupils in the larger technical high schools who are preparing for work in industry and commerce.) Now this examination, as a matter of educational policy, offers a very wide choice of subjects to candidates, so that in principle no normal pupil need select subjects that he dislikes or finds meaningless or too difficult. School organization will in fact restrict his choice very considerably, and perhaps in smaller schools reduce it almost to vanishing point. In the larger schools, however, a real choice is involved. The relative difficulty of the options becomes, then, a matter of importance. With the weaker pupils, this is all to the good. It is much better that they should make real progress in some subjects that appeal to them, even if these are non-academic, than that they should waste their efforts on unsuitable studies. It is unlikely that even the bare minimum in mathematics prescribed for these children will be seriously handled or adequately mastered, though that minimum is based on practical and sociological considerations, not on academic ones. Again, many stronger candidates who could and perhaps should master at least School Certificate mathematics will be tempted to accept the lower standard of their weaker classmates. Possibly some of our training college entrants illustrate this failure.

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[1] New Zealand Education Department, *The Education (Post-Primary Instruction) Regulations, 1945*, Wellington 1945.

It appears, therefore, that the students whose study of mathematics will be limited to the core will comprise two types—some abler ones who do not choose to go farther, and the weaker pupils to whom mastery of the core would in itself represent a creditable achievement. How will these children fare in their mathematical adventures?

Teachers are, by nature and force of circumstances, inveterate examination coaches. They ask for a clear and definite syllabus of work, a rigid prescription, a satisfactory textbook, and then, with Sarah Battle, 'the rigour of the game'. Pupils who are not preparing for examinations are neither fish nor fowl nor good red herring. So in the past, such non-university entrance courses as engineering, agriculture, commercial, woodwork, were treated with disdain. The pupils taking them commonly comprised an unscholarly group with limited intellectual powers. In the light of the old ideals there seemed no purpose in their schooling; yet all our present hopes are doomed if teachers continue to despise the weaklings.

It can hardly be denied that this danger still faces the non-examination pupil today. To be specific—what is the mathematics teacher to make of his core-limited class? They will admittedly never know any mathematics worth mentioning as such; they will have little interest in the subject, and will probably be very poorly grounded in its elements; they will never be faced by the threat of exposure in a public examination; they will require great patience and exceptional skill in the teacher who is to make any headway with them; they will leave school early, and there is no academic hope in them. Criticize this description as you like—call it exaggerated, unbalanced, unfair, too sweeping, too hopeless, too what-you-will. There are scores of secondary school teachers who in deed if not in actual words, declare that they have no interest in these pupils and hold no hope for their scholastic progress; and there are thousands of pupils who do as little core mathematics as they dare, and drop the subject at the first opportunity.

The way of escape for the teacher is not by resignation, suicide, or mass-murder. He can retain his sanity and self-respect, and indeed find a joy in his profession, if he faces the new educational situation calmly and sensibly. These weaker pupils are not to be despised because they are non-academic. Their very weakness is their claim for greater care and effort. Their lack of interest should be the stimulus for more realistic and meaningful teaching. Individual specialist teachers may have a right to be interested only in high scholarship, but the school as a whole must have teachers who are genuinely concerned with the welfare of slow learners—and even of non-learners, if such there be.

This demand is justifiable on several grounds. Ethically, it is dishonest, as it is certainly demoralizing, to accept position and pay for work that is not done, or only half-heartedly done. Socially and politically we are committed to a democratic respect for personality as such, and we are false to our traditions if we base service and respect on intellectual capacity alone. Even the weaker pupils will be full citizens; they may later be our economic and political masters. In any case as individuals in their own right they demand our best efforts. Psychologically, we are forced to realize that adequate living calls for more than intellectual acuteness, which may be associated with anti-social conduct and standards. Intellectual snobbery is no less objectionable and dangerous than any other. This is perhaps a hard saying for the young and enthusiastic mathematics master in his early professional career, and even for some mathematical experts who live in their ivory towers of abstract thought. It may be that the best master for weaker classes is normally the older and more experienced member of the staff, if he has not become fossilized: for he will have realized that life is much more complex than it appears to inexperienced youth, and that sympathy, patience and perseverance are pedagogically more important than efficient method and a caustic tongue. With these duller pupils we need a teacher who is not obsessed by academic standards, not examination-centred, not bound by tradition and convention and the desire for self-advancement, but entirely determined to get the best he can out of his pupils.

Such a teacher will not find the core mathematics course meaningless, or unworthy of his best efforts. Freedom from external examination is an inestimable boon to him. He is under no temptation to resort to cramming or to that undesirable forcing so strongly deprecated by the Consultative Committee set up to report on the New Zealand post-primary school curriculum. It explicitly recognized that while the prescription 'may be regarded as very elementary' it will extend the weaker pupils 'who will have difficulty in advancing beyond the very elementary stages in arithmetical computation'.[2] Indeed a secondary school principal, speaking recently at a mathematical conference, declared that in her school the 'two lowest forms did not, and could not, cover even the core'. The elementary stages however are of the greatest practical importance to the child. It is possible to live in the modern world with even less than an elementary knowledge of history and geography, of science, of art and music and literature; and indeed many people do so. But mastery of elementary calculation, of weighing, measuring, and estimating is required of every citizen, and the core prescription aims at securing this.

Many mathematicians will agree with the call for accuracy in mechanical operations, but contend that it is essentially the sphere of the primary school. I concur. But this does not mean that the secondary school teacher may ignore a faulty foundation on the simple plea of *tua culpa*. His task is to teach his pupils; and if they need training in mechanical skills, to give it. So we shall at least have a healthy attitude to work. Experience suggests that it will be necessary to give systematic training and practice in mechanical operations throughout the third and fourth forms; and I suggest that mental arithmetic is commonly too much neglected. A parent's comment on the arithmetic experience of his fourth form daughter is interesting in this connection. Himself a primary school headmaster, he complained that too little time was allotted to arithmetic, that 'mental' work was insufficiently utilized in this particular secondary school, and that the third form arithmetic was limited to mechanical work, so that the girl was allowed to forget percentages, profit and loss, simple interest, etc., which she had learned in her intermediate school. Such work had to be introduced to her afresh in the fourth form. I do not know how common this experience may be, but it certainly indicates an unnecessary source of weakness. Short but regular periods of mental work covering the field of Form II arithmetic would obviate this loss of previous knowledge, and maintain interest.

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[2] Consultative Committee, Report on *The Post-Primary School Curriculum*, 33. Wellington 1944.

The Report proffered the general advice that the course should be 'practical and the immediate usefulness of the exercises should be evident', and continued: 'There will be an extensive inter-relation of mathematics with other subjects, as, for example, in the use of numerical and graphical methods to solve problems arising in science, social studies, homecraft, and workshop activities.' The general soundness of these principles will not be disputed, though their specific application may be difficult. The appeal to interest is of obvious value. It should be noted, however, that in the particular form that it now takes this assumes an interest in science and economic conditions generally that will motivate an effort in arithmetic. Such an assumption may very well be wide of the mark. For example, the successful teaching of science to this group may be even more difficult than that of mathematics. The development of the two subjects *pari passu* and in close correlation, obviously highly desirable, is feasible though difficult. It is a matter of mutual dependence and aid. Each illuminates the other. It would seem that the legitimate and necessary demands of these applied subjects on the mathematician would cover the following: mechanical accuracy in arithmetical calculations, facility with percentages and proportion, familiarity with units of length, weight, etc., including metric measure, skill in handling formulae and simple equations, some ability to interpret and to construct graphs and other diagrammatic devices, neatness in working, precision of statement, and knowledge of simple geometrical truths and constructions. These, fortunately, can readily be taught even to weak classes, but not in a day, and it is hardly likely that they will be available when the scientist's need for them arises.

The growth of interest in mathematics will be found to depend ultimately on the ability of the pupil to master the given work without too great effort, and to feel that he is making headway. The success of the practical approach really depends on the fact that it tends to secure this effect, whereas the abstract approach simply baffles and confuses the less able children. Success in both science and mathematics teaching is intimately bound up with this feeling of increasing mastery. Most youngsters, indeed, find the appeal to later utility too weak to spur them to sustained effort. The future seems too remote to worry about. They live in the present, and work best where they succeed best. It is only when they feel themselves lost that they raise the cry, 'What's the good of this subject?'

My plea then, is that we recognize these non-examination pupils as important members of our classes, that we admit their scholastic limitations but regard these as increasing rather than diminishing the importance of our task, that we face the situation realistically and with the determination to do thoroughly the little that we can do, that we choose experienced, capable, and sympathetic teachers to handle them, and that these teachers be given freedom to interpret the core syllabus in their own way and time to work out that interpretation satisfactorily.

There is a most interesting field for investigation into ways and means of capturing the interest and developing the intellectual powers of the slow-learning group.[3] I have no authoritative or experimental evidence to offer on this problem, though some should soon be available.[4] Meantime, I would offer the following suggestions to the teachers concerned:

*Forget examination requirements and prescriptions, forget even the core syllabus, start with the class where you find it, and work at their most effective rate, not yours.

*While not expecting much progress along traditional abstract lines do expect real progress, for if nothing is expected nothing will be done. But progress will be rather in the direction of mastery of skills, than in the attainment of mathematical insight.

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[3] Two publications of the New Zealand Council for Educational Research have some bearing on this problem, the one directly the other indirectly: R. Winterbourn's *Educating Backward Children in New Zealand*, Wellington 1944, and G. W. Parkyn's *Children of High Intelligence: a New Zealand Study*, Wellington 1948.

[4] The experiment begun this year with such pupils at Kowhai Intermediate School may yield significant results.

*Hence do not be too fussily correct mathematically. The line between neatness, method and accuracy on the one hand, and meaningless form, petty restrictions, and factitious neatness may be difficult to define, but to the commonsense worker it is clear enough.

*Mathematics, here mainly arithmetic, is a calculating machine. Its aim is power, mastery, and (as far as possible) speed. Seek these. Other results are secondary.

*The encouragement of success is more potent than the spur of examination, the appeal of later utility, or the mere drive of a vigorous teacher.

*Forget your respect for mathematics—its symbolism, abstractness, generality, extreme logicality. Get the class working successfully, without fear and unshackled. One writer on the calculus prints as a sub-heading 'What one fool can do, another can'. Get that spirit into the class.

*Beware of abstruse and formal wording of problems. Too often, the greatest difficulty for the pupil is to know what the question really means. It is singularly unpleasant to be required to use a method one does not understand to solve a problem that one understands even less.

*If some sections of the prescription prove to be difficult or meaningless (the fault may be yours!) leave them out. It is better that your pupils retain their confidence and limited power than that they flounder in difficulties. If you cannot swim more than a few strokes, play round within your depth. You can have much fun, and may even come in time to swim further. If you are barely rescued from drowning in your first attempts, you will soon desert the water altogether.

One practical difficulty will always confront the teacher. He cannot be sure that some of the pupils who are taking only core mathematics will not need this subject later for the School Certificate Examination. Indeed, I am informed that this doubt does in fact sometimes cause too many pupils to attempt the harder course, and that this in its turn forces those who are successful to continue the subject in post-Certificate classes. But in the group we are considering there can be very few of these. In any case we must be realists, and it is no use 'moving on' before the class is ready to advance. Officers of the Education Department look for sound work rather than a specious completion of a syllabus.

From experienced teachers of core mathematics, I have been given the following suggestions:

There should be, in teachers' training colleges, lecturers with a very special interest in and knowledge of arithmetic. They could presumably be in close touch with textbooks and practices of both primary and post-primary schools, and with work in other countries. An important part of their work could very well be the provision of liaison between the two branches of education, giving direction to the arithmetic of primary schools, and ensuring that post-primary schools are familiar with work and methods of primary schools.

Every head of the mathematics department should take at least one core class (preferably a different one each year) for some years, until he is able to give practical advice to junior members of the mathematical staff. This of course assumes that he is given time to hand on to others the benefit of his experience and training, but in fact he is frequently too busy to do so.

An authorized core mathematics textbook should be published as a guide to New Zealand teachers.[5] A qualified and experienced mathematics teacher should be set free to write such a book: it cannot be written in spare time and school holidays. He will obviously need the assistance and co-operation of others. (Some teachers, however, feel that the time is not ripe for this. They think that we are not yet sufficiently sure of what we want or of what can be done, and fear to limit or discourage individual experiment at this stage. That a standard textbook will ultimately be required is admitted.)

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[5] Since this was written a Committee has actually been set up for this purpose.

Due consideration must be given to the varying needs of the two sexes. In smaller and mixed secondary schools this may be specially difficult. For example, a class composed of 'commercial' girls, and boys preparing for trades is an unsatisfactory combination.

The time to be given to mathematics with these slow learners is an important consideration. The Education Department's Regulations set out certain minima for all classes in all subjects, but it is obvious that with weaker forms these times must be greatly exceeded. Indeed, I suggest that these forms will need almost the full twenty-five hours per week for the core alone. Internal organization may make this impracticable, and heads may regard it as undesirable, but I propose to work on the assumption that times in all subjects have been increased proportionately to make up the total week.

Under these circumstances, we should have an allowance of some six hours per week for 'general science and mathematics'. The apportionment between these two subjects is not specified, but an indication of the official attitude may be gained from an examination of the respective prescriptions. My own general impression is that while the mathematics prescription has been drastically restricted the science course has been generously and even recklessly enlarged. This springs, I believe, from an over-estimation of the possible effects on these children of a study of science in general—whether the physical and biological sciences or the social studies. Conversely, the possibilities of a reformed mathematical training appear to me to be underestimated. We must, of course, allow for the general tendency of syllabus framers to use high-sounding terms to represent very modest content. Certainly, all that these pupils will assimilate from this science course will be very elementary and one may anticipate some disappointment in the final outcome. This does not directly concern the present investigation, but it raises two problems. Firstly, the learning capacities of these children are severely limited, and any over-pressure in one subject may unduly depress performance in other subjects. Secondly, the time given to mathematics may be correspondingly reduced.

In all subjects, with these pupils, we must be conscious of their limited ability, the importance of thoroughness in essentials, and the seriousness of our aims. I suggest, therefore, that our guiding principle should be: the essential minimum, thoroughly mastered. What is essential? The mathematician, thinking of his subject as a systematic co-ordinated science tends to include much that is important only as part of such an organic whole. This will not do. Even our syllabus framers, who may be suspected of underestimating the significance of mathematics in life, are apt to stress topics that are non-essential, especially in the realm of social and economic experiences. The pupils with whom we are now dealing will not be long at secondary school, and their studies will cease with their schooldays. Half-mastered skills will soon be lost and memorized facts may soon be confused or forgotten, or rendered dangerous by changing conditions. Thorough mastery of the mechanics of calculation and reasonable speed and accuracy in straightforward calculations are fundamental. Where these have not been secured in the primary school the last hope lies in the secondary school.

§2 SCHOOL CERTIFICATE MATHEMATICS

The Consultative Committee's report recognizes a first group of 'perhaps 20 per cent of post-primary pupils [who] will require a course that leads up to university mathematics or science, or advanced technological work. The second group, approximately 40 per cent, could with advantage to themselves study the subject up to the present [i.e. old] University Entrance level.'[6] These pupils are obviously quite capable of sustained and serious study, though they should not aim at later specialized university work, in mathematics at any rate. They comprise a group of children definitely above the general average of 'intelligence' of the whole community, having a median IQ of 105–110. In other words, they are able pupils, but not 'scholars' in the narrow sense of the term—they are children with probably much practical and social intelligence, and with varied abilities. Such people form a solid capable group in any community and we neglect them at our peril. For this group the School Certificate is regarded as a satisfactory examination goal. Three major weaknesses in the present teaching of mathematics are listed as follows: '(1) In algebra, the traditional approach, still widely used, is destructive of interest. Pupils are kept too long on complicated examples in the elementary stages. (2) Aids to calculation are neglected too long. For example, the use of tables (ready reckoners) and logarithms at an early stage gives that realistic flavour to arithmetic that encourages accuracy and maintains interest.

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[6] *The Post-primary School Curriculum*, 50. Without any authority from the Committee or anyone else, I should venture to interpret this to mean, under the new conditions, 40% to School Certificate standard; 15% to University Entrance standard or perhaps a school year beyond; 5% to scholarship (credit) standard. These percentages appear to me rather hopeful: they could perhaps be realized, but the appeal of less exacting options will probably reduce the first two percentages.

(3) In the teaching of geometry there is need for considerable change.' (The Committee obviously implies that the change should be in the direction indicated in the reports on geometry issued by the British Mathematical Association.[7]) An effective study of geometry is regarded as beginning best through the use of drawing instruments and scissors, and the practice of writing out and memorizing theorems is adversely criticized. 'To ask young pupils to present the usual formalized proofs is to overburden their memories and to neglect the opportunities offered for fundamental training in logical progression.' Hence the proposed prescription 'is designed to encourage the modern, realistic approach we have in mind.'[8]

The examination prescription set out in the Education Department's Regulations follows the suggested course very closely. A few verbal changes are almost the only modifications. It assumes a three- or preferably four-year course by such pupils. Some of the new features, designed to remove the weaknesses listed above, may be mentioned.

The introduction of statistical work, in the form of graphs and simple tabulations, reflects the double aim of making elementary mathematics of practical everyday use at every stage, and of ensuring that pupils will be able to follow sociological problems in after-school days. Statistical work makes possible the early application of mathematics to everyday problems of all sorts, so killing two birds with one stone: it teaches mathematical methods and concepts, and at the same time brings the pupil into contact with information in science, social studies, everyday life. A pupil who leaves after even one year of secondary school mathematics should, in this respect, have learned something of value.

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[7] Mathematical Association, first report on *The Teaching of Geometry*, 1923, and second report 1938.

[8] Consultative Committee, op. cit., 52.

Just how far this work should go at the School Certificate stage, however, is doubtful. While some introduction to very elementary pictorial, diagrammatic, and graphical work is possible and desirable with core pupils, the further elaboration of this work raises ticklish questions. Statistical work is deceptive and dangerous, full of snares for the ignorant and the unwary; and superficial knowledge may expose its owner to dangers that do not exist for the completely ignorant. (At various stages in mathematics it is possible to offer rigid proofs of certain fallacies—for example, that 1 = 2, or that all triangles are equilateral—which at earlier stages are meaningless and at later stages can be refuted.) For this reason, some experienced teachers would approve of the treatment of statistics as set out in the core prescription, but would hold over further treatment till the sixth form, and there handle it thoroughly.[9] In any case, the introduction of the work depends largely for its justification on the use made of it by social studies, science, and geography teachers in their ordinary classroom work. This is its natural milieu; elsewhere it fits awkwardly into existing mathematical schemes.

The greater stress on graphical work is now accepted everywhere. In its very simplest form, it provides a useful and practical exercise in interpreting data and conversely in putting meaning into the graphs and diagrams now so commonly found in books, journals, and newspapers. This latter aspect is of obvious importance. It also leads to an easier

[9] cf. F. W. Westaway, *Craftsmanship in the Teaching of Elementary Mathematics*, 538. London 1931. Gt. Britain, Board of Education, Educational Pamphlet No. 114, *The Organization and Curriculum of Sixth Forms in Secondary Schools*, Chapter VIII, H.M.S.O., London 1938. Mathematical Association, report on *The Teaching of Arithmetic in Schools*, 75, London 1932.

understanding of the idea of function and variation, so important in later mathematical thought. The solution of every-day problems of changes in speed and direction without artificial limitations of assumed uniformity is of particular value in these days when we are so much concerned with motion. The chapter on graphs in Durell's *School Certificate Algebra* should be thoroughly mastered; the book then needs supplementing only for the ideas of gradient, increment, and intercept. It should be unnecessary to add that graphical work must not be treated as a water-tight compartment. Such a treatment would be contrary to the whole spirit and purpose of mathematics. The Education Department's pamphlet on *Statistics*, (Post-primary School Bulletin Vol. 2, No. 2), is interesting in this connection. Its separate publication inevitably tends to suggest the uniqueness of statistics. Indeed, while both the opening and the penultimate paragraphs stress its bearing on science, economics, and social studies, there is throughout the booklet an absence of direct linkage with the usual branches of mathematics. This is the more regrettable in that many of the more intransigent teachers of mathematics oppose the introduction of statistics as being a dead end, an excrescence on the body of legitimate mathematics. There is in fact no very obvious reason why simple statistical methods should not be part and parcel of the standard introduction to mathematics. The graphical approach to algebra, for instance, and the frequent handling of graphs throughout gives ample scope for statistical treatment. Likewise, the calculation of averages in arithmetic may readily be extended to the finding of medians and quartiles, and the advantages and disadvantages of these different measures can be simply illustrated. But the thorough and healthy incorporation of some statistical methods into the normal structure of secondary school mathematics has still to be achieved by some qualified writer of a textbook on mathematics.

On the whole, it can be claimed that there is little to cavil at in the syllabus for Paper A, and that it provides the opportunity for work and examination along approved lines.

Paper B (geometry and trigonometry) calls for particularly careful interpretation of the syllabus by the teacher. The enlarged syllabus in trigonometry should not cause undue worry to the teacher, particularly if easy numerical trigonometry has been added to the third form syllabus. Much of this work, properly approached, is very simple, and its immediate applicability to everyday problems adds greatly to its interest. We have had this subject with us in the University Entrance Examination for some years now, and teachers have worked out satisfactory methods of approach. The days are over when trigonometry was a sixth form subject, and even then most confusing for the first few months. Its early introduction in a more practical form, and associated with graphical work or similar triangles, is a great advance. The work may still be presented in too formal and abstract a manner, but this danger is relatively small. It is still seen, however, in the overstressing of identities. Simple solution of right-angled triangles can be taught practically, without appeal to theory, and varied exercises add greatly to zest and confidence. Such work can even form an introduction to the use of tables of trigonometrical values—without at this stage having recourse to the confusing differences tables.

The new syllabus puts more stress on mensuration as applied to solid figures than was the case in the old university entrance work. This is in keeping with the modern view that school mathematics has too long been restricted to plane figures, whereas three-dimensional figures are the most common experiences in life. This limitation reflected the old educational theory that plane geometry was simpler than solid, and should therefore precede it. Hence, as in Euclid, solid geometry was reserved for more advanced pupils. The belief that solid geometry is inherently more difficult than plane geometry springs, however, largely from the more fundamental insistence on the abstract treatment of the subject. Many of the mathematical properties of solids can be grasped quite easily by young pupils if actual models are used. These may be either genuine solids or skeleton models made of wire, but in either case they should be made with precision. Both types have their advantages, and they might well be used as complementary aids. Many post-primary teachers are today experimenting with such models, and their results, generally speaking, are very successful. In those courses which prepare pupils for industry and trade and which make use of tools, instruments, and mechanical equipment, the technical high schools appear to have an advantage, and their pupils are well placed to get a sound grasp of geometrical principles. All schools, however, can make intelligent use of models and similar teaching aids.

One of the main difficulties in teaching solid geometry from books is connected with the representation of solids on paper. Children have trouble both in representing a solid on a plane surface and in visualising a solid from a given diagram. A definite course of instruction in three-dimensional drawing is advisable. This may or may not be done in association with the construction of solids. An excellent model lesson is ready made for the teacher in a report of the British Mathematical Association. Few teachers would fail to benefit from having taken this lesson with a class.[10]

It should also be noted that the correlation of the Paper A Syllabus with that of Paper B would be helped by extending the trigonometry syllabus to include three-dimensional examples. This practice is apparently not barred even now, as the 1948 paper included such a problem. More numerical three-dimensional exercises are available in trigonometry than in geometry.

How geometry may best be treated is a more perplexing question. This subject has traditionally been presented in a formal and logical manner, its practical bearing on everyday problems being ignored or confused. The subject raises the vexed questions of the nature of proof and the relative importance of theorems and of riders. The new textbooks (e.g. Durell's) cannot be followed so blindly and mechanically as the old, and some understanding of the educational principles underlying them is essential. Mathematics teachers would do well to familiarize themselves with the two reports on the teaching of geometry, issued by the British Mathematical Association. These reports[11] recognize an *experimental and intuitive initial stage* in learning geometry. This calls for practical work with solids and plane figures, allows the apparently obvious to be accepted, e.g. that vertically opposite angles formed by two intersecting straight lines are equal, and may include simple deductive work (mainly oral). At this stage the pupil becomes familiar with the subject-matter but is led to develop precision in the use of technical terms and the expression of his own arguments. The danger that it may become desultory and aimless

[10] British Mathematical Association, *The Teaching of Geometry in Schools*, 143. Second Report, London 1938. This should be preceded by a careful study of pp. 35-43, and particularly of pp. 84-90.

[11] *The Teaching of Geometry in Schools*, First Report, Third Edition 1929. Second Report 1938.

is recognized. The second stage, the *deductive stage*, covers a period of three years. The whole of elementary plane geometry is to be covered, with 'some inroads into solid geometry'. Several groups of theorems, important because of their implications, should be known. 'The reasoning powers are now strong enough for deduction.' At the end of this period a boy 'should know the interesting theorems of plane geometry, he should be able to devise and solve easy riders; he should be able to apply his knowledge to simple solid figures; he should have some grasp of logical method. On the other hand, the theorems he knows will not at present be deduced from a very small number of axioms; his knowledge will rather be based on a considerable number of assumptions, assumptions which are appropriate to his present mental development.' The first Report notes that 'There is no reason why an individual proposition should not be known long before any attempt is made to deal systematically with its subject matter, and the rider that undertakes an incursion into territory where there is no immediate prospect of exploring thoroughly is most stimulating. In particular, details of solid geometry should become familiar far in advance of any systematic abstract study.'[12] A third stage is known as the *consolidating stage*, and described as the 'organization of derived propositions'. In it pupils learn that the whole of their geometrical knowledge is logically connected together. More rigorous proof is required. The abler boys need this rounding off and consolidating. A fourth stage, the *systematizing stage*, is described as the 'organization of primitive propositions', and is suitable for the first year of sixth form work.

The first three of these stages, if interpreted by a competent teacher of mathematics, cover the syllabus that we have adopted for Certificate purposes; but there is nothing in the prescription to make this explicit, and the unfortunate untrained teacher with little mathematical skill completely fails to realize it. What I regard as important is the recognition of these stages in teaching practice. The 'consolidation' of the third stage should not be attempted before the second stage has been thoroughly mastered. This is a matter of performance, not of age. The main danger is that haste to prepare pupils for the examination will cause a scamping of the free and natural methods of stage two.[13] The style of presentation in stage three is described as 'approximately Euclidean, but not Euclid'—because Euclid's style is too prolix. There is no need to memorize the numbers of theorems; the use of hypothetical constructions is admitted; attention is paid to the grouping of propositions; every effort is made to generalize results, a modern and highly important characteristic; and the synthetic form of Euclid's proofs is recognized as insufficient. In teaching, analysis is necessary: the neglect of this aspect of proof has been a fundamental weakness in our teaching. Solid geometry, traditionally neglected because of its supposed difficulty and for lack of time, should be included. The use of modelling in paper and cardboard is also discussed in this report. An article by E. G. Hasell on the teaching of geometry in the *New Zealand Education Gazette*, 1 May 1945, accepts the above recommendations almost completely, and may help readers who cannot refer to the Mathematical Association's reports. Much care should be taken to ensure that some overlap of practice occurs in the various stages—they are not to be too sharply defined. This is crucial to the whole scheme.

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[12] First Report, 12.

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[13] Details of these stages are set out on pp. 17-25 of the First Report and on pp. 113-7 in the Second Report.

A few comments on the new syllabus and its official justification may now be added.

The Consultative Committee's Report. Experienced teachers who have taught successfully on traditional lines will want something other than a bald statement that their methods are destructive of interest to convince them, but even the most conservative teacher must be impressed by the considered opinions of a group of experienced teachers and educationists such as formed the Committee of the Mathematical Association. And now we have the work of van Zyl,[14] with an excellent bibliography from which the Education Department could get plenty of ammunition. All teachers of mathematics should read these books.

Algebra and Arithmetic. Paper A puts more stress on mensuration, especially as applied to solid figures, than was the case in the old University Entrance papers. The introduction of the mid-ordinate rule for finding an approximate area is a step forward, but it appears to be too specific. It is at least arguable that the trapezoidal rule is just as easy. Some technical schools prefer Simpson's rule, which is easy to apply and is usually more accurate than the mid-ordinate. Verbal memorization of this rule is not desirable, though some textbooks suggest otherwise, offering a statement that looks quite formidable and repulsive. The important development here is the introduction of approximate methods where complete accuracy is unattainable. The fundamental idea of the need for approximation is important, not any particular method of calculation.

A matter of prime importance is the estimation of the extent of accuracy that can be obtained from measured quantities. As van Zyl points out, many teachers themselves

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[14] *Mathematics at the Crossroads.*

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do not know enough about this. I suggest that the syllabus should include the treatment of significant figures; approximation; measured quantities; the determination of the maximum number of reliable figures in the sum, difference, product, and quotient of measured numbers, and the constant use of this number in all calculations involving measured quantities.[15]

Geometry and Trigonometry. The School Certificate syllabus in geometry gives teachers the opportunity of spending the first year on the work outlined above. However in view of the close agreement on the method of presentation and on the content of geometry between the Mathematical Association's report and the *New Zealand Education Gazette* article, it is perhaps a pity that the syllabus has not followed more closely along the lines of Chapter V of the report. In this Chapter theorems are classified as follows:

(a) Theorems in which the deductive process helps to introduce the pupil to new facts; the proof is here, in fact, the best way of convincing the pupil of the truth of the theorem. The pupil should work through them carefully and be able to reproduce the proofs.

(b) Theorems whose results become apparent more by symmetry or intuition than by proof. Pupils should be able to prove these as easy examples of the deductive process, but not be expected to learn the proofs.

(c) Theorems whose results are important but whose proofs should not be expected to be learned at Stage B owing to their difficulty.

It seems reasonable to expect the proofs in (a) and (c) to appear in a School Certificate syllabus. The double grouping is of course more a teaching direction than a syllabus one. Those that are in (b) would not be required for examination purposes. There appears to be some doubt about those dealing with parallelograms; I would put them in (b).

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[15] cf. *Mathematics at the Crossroads*, 58-65; *The Teaching of Arithmetic in Schools*, 66-9.

Looking at the syllabus from the point of view of the above classification one could make the following criticisms and suggestions:

(1) The syllabus at times is not sufficiently specific: e.g. 'relations between arcs, angles, and chords', 'relation between chord and centre', and 'tangents and common tangents' are all parts of the syllabus which contain or can be made to contain several theorems of the (b) class, which should be omitted. It would have been an improvement if the list of theorems required from this portion of the syllabus (severely pruned) had been stated.

(2) It is surprising to find the 'key' theorem of area work omitted. The equivalence of all parallelograms on the same base and between the same parallels is surely not obvious.

(3) 'The intercepts cut . . . straight lines' would have been better if given as 'The mid-point theorem, the intercept theorem, and the general intercept theorem' or as 'The mid-point theorem, and the intercept theorem'.

(4) The 'difference of squares locus' might profitably be included. It is a most useful theorem with many time-saving applications and is satisfying as a locus theorem. The only textbook that I know that specifically mentions it and uses it is Forder's;[16] it is mentioned in the Second Report, p. 57. Professor R. J. T. Bell of the University of Otago used it consistently. It should be better known. (See the proof for concurrency of altitudes by means of it.)

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[16] H. G. Forder, *School Geometry*, London 1948.

(5) Trigonometry problems should include three-dimensional ones.

(6) The desirability of prescribing 'simple identities' at this stage may be queried. It may easily lead to that juggling with forms that was so unhappily characteristic of the older approach to algebra.

Finally, it may be affirmed with some confidence that the new syllabus, interpreted correctly and taught efficiently, represents a very great improvement on the old-style course. It does not call for outstanding ability yet it represents a useful and living treatment of the subject. It deserves the best efforts and fullest co-operation of all teachers.

§3 SIXTH FORM MATHEMATICS

We pass here to a realm where the emphasis in teaching shifts from pupil to subject. Hitherto, while seeking to modify the content and to improve the methods of teaching mathematics in our schools, we have kept an anxious eye on the mental limitations and practical needs of the pupils. We have affirmed that, whatever the value of abstract mathematics to the gifted few, many pupils have not grasped the principles of the subject, and so have found it a meaningless and purposeless study, remote from life. We have urged that much of this frustration springs from the unsatisfactoriness of the traditional methods and that a different approach might enable such students to study the subject with both interest and profit. We have thought of the pupil as a more or less average human being to be trained to take his place in society, and of the subject as a means to this end. The economic, industrial, and psychological needs of the pupil have loomed large; and mathematics has been regarded mainly as a useful means to securing non-mathematical ends.

This attitude obviously tends to lead to the opposite error—that we may fail to do justice to the abler student—and this charge is frequently made against the 'new education' itself. Reeve[17] writes of conditions in the United States: 'The most retarded pupil in the secondary school today is the gifted pupil, the one with a scholarly mind. The secondary school machine is all geared up to turn out a mediocre product'. A girls' grammar school mistress considers this to be 'especially true in a girls' school where about one third of IIIA are really first class'. Similar warnings might be quoted from many writers, and cannot be ignored. Democracies must learn to utilize the rare powers of genius as well as to socialize the average and to shelter the subnormal. And these diversified duties are not easily borne in mind at the same time.

To avoid all possible misunderstanding, therefore, I affirm here explicitly and unequivocally the rights of our ablest pupils to treatment in accordance with their powers. To restrict the clever third-formers to the core syllabus and its associated methods would be absurd and disastrous. In current practice, this is not done. Classes taking core mathematics and School Certificate and University Entrance mathematics are usually sharply defined, and the clever boy is very unlikely (in all but the smallest schools) to be in a core mathematics class pure and simple. School Certificate and University Entrance prescriptions are usually dealt with in a way that does no injustice to him. Though School Certificate requirements do cover the core, it is only in the purely core classes that the treatment is likely to be unsuited to the scholarly type. It is true, however, that the gifted

[17] W. D. Reeve, 'Coordinating High School and College Mathematics' in the *Mathematics Teacher*, 354–64, December 1946.

pupil is not normally segregated and pushed on at a rapid pace, as may be the case in England.(For myself, I would not regard this as desirable, even if it were possible.) I note with approval the warning, 'We accept the need for and value of specialisation, but the human being should come before the specialist and never be lost in the specialist'.[18]

The secondary schools must always cater for the intellectuals; and they will not readily neglect this duty. The difficulties of their task may be increased by the new type of entrant, the core requirements, the varied extra-curricular activities, shortage of staffing and the increasing complexity of timetables and general organization. On the other hand, the mathematics teacher is likely to be freed, beyond the core, or at any rate School Certificate standard, from a considerable number of those pupils who in the past tended to hold up the progress of his class. The work of the best students in University Entrance Scholarship mathematics is today of a high standard—definitely higher, I believe, than it was a few years ago. This is presumably due in part to the new emphasis placed by schools on science and mathematics, including a strong tendency in boys' schools to encourage the ablest pupils to select these subjects for the University Entrance or Scholarship examinations, the new concentration on mathematics, pure, 'additional', and applied, and perhaps to more successful teaching methods. During the period 1939-48 an average of 14 candidates annually presented all three options in mathematics (1400 out of 2400 marks), usually along with physics and another science. Scholarship winners averaged 6 per year taking all three options. From a purely educational standpoint, such limitation of studies must raise serious misgivings. Even if the specialization does not occur till Form VI, it means that clever children from the age of say 16 years are largely cut off from the more general studies, cultural and humanistic, before they have reached a social maturity. This must surely result in a serious lack of balance in character and outlook.

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[18] Council for Curriculum Reform, interim report on *The Content of Education*, 51. London 1945. See also G. W. Parkyn, *Children of High Intelligence*, 167-70.

I would stress first, though it cannot be discussed here, the effect on the sixth form of the new organization of secondary schools, consequent on the new system of options. At this stage, the few but extremely important survivors from various courses are united. Normally, therefore, the form will comprise a varied assortment of young people with different aims and backgrounds. Except in the largest schools the individual groups will be small, not justifying teaching as separate groups, however much that may be needed. The master, therefore, must develop the technique of handling two or more groups effectively. This is not really difficult, but it does require thought and organization. To cite a useful statement of the Board of Education: 'The relative importance of group instruction and individual work will need to be weighed. Both have their place. It is desirable that pupils should come into intellectual contact with masters and with each other in a way that only oral work can provide. It is equally important that they should learn the valuable lesson of working for themselves. This latter training is dictated by necessity in the smaller schools, but its importance is perhaps overlooked sometimes in the larger.'[19] My own teaching experience in small secondary schools has impressed on me not only the possibility of gaining scholarships under such apparently adverse conditions, but the actual advantage of being prevented from teaching too much—the beginner's typical fault. At a lower level, I have noted similarly that the success of the Education Department's Correspondence School has important lessons for us on this matter of over-teaching. This school has inevitably a high percentage of failures, for many of its pupils work under most unfavourable home circumstances. Yet I found, during my association with the school, secondary pupils making excellent progress in specifically post-primary subjects (Latin, French, mathematics, science) with only typed assignments to guide them and model answers plus teachers' notes to help them to correct their written work.

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[19] Board of Education, Education Pamphlet No. 114. *Organization and Curriculum of Sixth Forms in Secondary Schools*, 50.

The prevailing conditions in general are these. Some pupils, not proceeding to the university, continue with the subject because they like it, or from Hobson's choice as they lack sufficient other subjects to satisfy the requirements of the University Entrance Examination or of the principal. These may be mediocre pupils who have gained a false impression of their powers from a fairly good pass (say 70 per cent) in the School Certificate Examination. Such a mark may result from particularly good teaching or from successful cramming, and so not represent sufficient ability for specialization in the sixth form. Others may plan to go on to the university, where they will continue the study of mathematics as a pre-requisite to an honours degree in science, or to professional engineering, actuarial work, and the like, or for an honours degree in mathematics itself. Such students will comprise some who are satisfied with the University Entrance qualification, some who seek bursaries or scholarships —the latter including some who take 'additional' mathematics and mechanics.

Dealing with similar conditions in England, the report on the Higher School Certificate Examination advocates three types of examination:[20]

A. Advanced Mathematics, catering for third-year sixth form mathematical specialists;

B. Mathematics (i) pure, (ii) applied, for second-year mathematical specialists; and

C. Mathematics for science candidates.

In New Zealand the corresponding groups of pupils are scattered through boys' or girls' secondary (grammar or high) schools, private colleges, mixed high schools, technical colleges, and district high schools. What they are given in mathematics consequently varies greatly.

Class teachers of sixth form mathematics are on the whole well qualified academically for their exacting work. In some girls' schools and in smaller high schools, it is difficult to secure specialists, but their needs are now being recognized. The award of teachers' bursaries to sixth form leavers, enabling them to complete university degrees on condition that they subsequently enter a teachers' training college and later teach for a certain time, makes it at least possible for the Education Department to redress in some measure the balance between arts and science degrees and between demand and supply of specialists in mathematics. This will soon have its influence on the schools.[21]

Possibly our specialist teachers of mathematics have too limited an interest in the broader educational issues of their teaching, in the sixth form no less than elsewhere. In this matter, English mathematicians such as Nunn, Whitehead, and Jeffery, have given us a lead. The suggestions made by the New Zealand Secondary Schools' Association and the Auckland Mathematical Association indicate sincere and lively interest in the minutiae of classroom teaching and syllabus requirements, but broader reference to industry on the one hand and to the psychological needs of children on the other is conspicuously lacking. There is probably too little variety in our teaching methods and outlook. No evidence of special consideration appears to have been given to the case of a pupil who is not taking the subject past sixth form standard. Should, for example, 'statistics' be further developed, in conjunction with other branches of mathematics, for such a student? Ought provision to be made for descriptive geometry, commercial arithmetic, etc.? The larger technical schools of course do make special provision for trade and professional examinations.

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[20] Gt. Britain, Board of Education, report on *The Higher School Certificate Examination*, 82. H.M.S.O., London 1937.

[21] As a separate issue, not to be pursued here, one may question the long-term advisability of this pre-selection of entrants to a profession. It may obviously lead to a competitive scheme of bursaries for various professions which may or may not be socially desirable.

With the content of the course, teachers appear to be reasonably satisfied. They have taken their part in drawing up prescriptions and in modifying them from time to time. It is possible that they have too readily acquiesced in meeting the university's requirements, provided that these could be satisfied under prevailing conditions. At any rate, I have not heard of any radical changes being advocated. The work as now carried out appears, therefore, to be reasonable in both content and appeal. A normally intelligent sixth former generally contemplates with considerable pleasure the new and interesting vistas which open out before him, often with dramatic suddenness, at this stage where he finds himself a member of an ever-narrowing band of enthusiasts. But a syllabus still more adapted to his particular needs could perhaps be devised.

The treatment of sixth form mathematics tends rather too much towards abstractness. More help, for example, should be given in geometry by means of solid models and frameworks, and the question of change in the nature of the syllabus in solid geometry (similar to that made in the School Certificate syllabus) is worthy of consideration. This latter point, it may be mentioned, has already received some consideration by the Auckland Mathematical Association. The syllabus would be much less 'stodgy' if many of the theorems were assumed intuitively and applied immediately to more valuable territory.[22] Also the use, for example, of the 'Difference of Squares Locus' provides more interesting, and neater, methods of dealing with much of the work on perpendicularity. The lack of suitable textbooks in algebra and geometry is claimed by many teachers to be a drawback. The sixth form stage should not require the grouping together of sections from different textbooks. The demands of trigonometry, arithmetic, mechanics, analytical geometry and calculus appear, however, to be adequately dealt with. The algebra of 'Additional Mathematics' appears to require particular care in both treatment and in selection of examples.[23]

There may be some lack of co-ordination between syllabuses for the University Entrance and the University Scholarship Examinations, due to the different consideration received by each and the time-lag before recommendations come into effect. Vectors, for example, are always set in the University Entrance Examination, but not in the Scholarship Examination, though there is opportunity at times for their use in Scholarship papers. This lack of consistency, however, may be more apparent than real.

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[22] See H. G. Forder, *Higher Course Geometry*, 1931.

[23] *The Higher School Certificate Examination*, 81-4. Chapters 1 to 5 give further useful information.

With the large number of options at present available, many schools can only with difficulty keep separate their first- and second-year sixth form classes—and then perhaps for only part of the time. This fact, together with the large range of abilities in classes taking School Certificate options, tends to hold back the brighter pupil, and to leave the less gifted one stranded. The preparation of a course particularly suited to the able but non-academic boy appears to merit consideration by all teachers; a radical difference in content and treatment might appear advisable. The brighter boy, if he is to receive the attention he deserves, might possibly be transferred to one of a series of central schools each specializing in a definite range of subjects. Those schools specializing on the mathematics side would naturally include science subjects in their sixth form syllabus and would be in a position to give full courses in mathematics, additional mathematics, and mechanics. (On the other hand schools specializing in science would not necessarily have mathematics courses of the same content. Mechanics, for instance, could not profitably be used as a 'cloak', for some of the more difficult problems belong by right to other branches of the subject.) Special provision could then be made in most post-primary schools for those sixth-form pupils not going on to higher education. Selection for such advanced courses would be partly, but only partly, based on very good results in the School Certificate Examination, and special consideration could be given to those who would benefit by advanced work at one of the larger technical colleges. Encouragement, and guidance, would play a large part in the direction of suitable candidates. There would not then be the need for conscientious teachers to saddle themselves with additional burdens in an effort to give good pupils a chance. Such advanced courses could be instituted in all our principal centres. Different university courses at the different university colleges may accentuate local modifications in the treatment of sixth form mathematics.

There are, of course, many obvious and very serious objections to such a proposal. Even if it could be shown beyond all doubt to be advisable as a step towards raising standards of work, this would still have to be weighed against its effects on those schools that would lose their brightest seniors. Even the final welfare of the pupil himself might be prejudiced by the transfer. I merely suggest the possibility of such a course, and note that district high schools already lose their seniors in this way through bursaries to city schools. A further very serious objection is the introduction of still another break in schooling. The bright pupil might well find himself attending in turn a primary, an intermediate, a post-primary, a specialized post-primary school, and then a university college. To preserve the unity of the educational process throughout would puzzle even the master-mind. Why are social problems so complex?

I add a few remarks on the requirements of the university, for they affect the post-primary schools very materially.

The emphasis in university mathematics is on the subject rather than on the student. The professor seeks to teach John mathematics, but in the last resort he is a mathematician rather than a general educator. In each of the university colleges serious and sustained efforts are made to help students who for one reason or another are ill-prepared to begin Stage I mathematics. But ultimately the student must measure up to the examination standard. He attends classes for that very purpose. The university exists to provide higher education for those who want it and are capable of profiting by it. The university or technological college has therefore not only the right but the duty to decide standards both of entrance to and graduation from its classes.

In passing it should be noted that this favoured position of the university tends to create in the minds of professors a completely misleading idea of normal standards, and consequently a serious misunderstanding of the problems of the class teacher. It is known, but too often not realized, that even the poorest students are usually well above the general average of the community in their academic or abstract intelligence. As examiners for university entrance, professors do see the work of relatively weak (not 'accredited') students: but even the children who fail here are in fact often far above the average level. A secondary headmaster and former inspector declares that the 'standard of the average pupil is of a badness incredible to the university staff, who see only the best'. It is this fact that makes university criticism of secondary school methods and achievement sometimes valueless to the extent of absurdity, and that made the domination of schools by the old University Entrance Examination such a disaster. If the university is entitled to call the tune in higher education, it is by its very nature disqualified from guiding and controlling the education of the non-academic pupils. Yet in fact, by its requirements it can and still does indirectly affect the nature of courses offered by secondary schools.

Theoretically, entrants to first-year mathematics at the university may comprise four groups of students: those who have taken the subject to core standard only, those who have taken it to the School Certificate standard, those who have taken it to University Entrance Examination standard, and those who have taken it to Scholarship standard (with or without Additional Mathematics).

It is evident that the first two groups are insufficiently equipped; and if, for some unexpected reason, they must later take the university class, they will need long and intensive preliminary coaching. The number of such students should, however, be very small. Secondary school principals realize that ignorance of mathematics is a hopeless handicap for a science degree, and will seek to direct abler pupils accordingly. Indeed, in the bigger boys' grammar schools it is now the admitted policy to encourage the ablest youngsters to specialize on the science-mathematics side. Those who drop mathematics early are more likely to be girls; but even these will almost certainly have taken School Certificate mathematics, and this, though limiting, will enable them to take Arts courses. The district and country high schools are liable to find trouble in securing specialist teachers of mathematics and are likely to be forced in consequence to present pupils in other subjects at the higher levels. The new regulations requiring country service from secondary school teachers may help them—but not very much.

The prescriptions for University Entrance and Scholarship mathematics have recently been examined by an Auckland Committee set up for the purpose. Apart from various detailed suggestions which need not be quoted here, this Committee reported as follows:

1. That the present University Entrance Examination represents a reasonable standard of attainment for the average pupil in his first year in a sixth form.

2. That for those not taking mathematics, physics, or chemistry at the University's Stage I level, the pass standard of University Entrance mathematics is sufficient.
3. That for those who propose taking Stage I mathematics, physics, or chemistry the pass standard of University Entrance mathematics is not sufficiently high, and that those Entrance candidates intending to take Stage I mathematics, physics or chemistry be advised to obtain a 'good pass'—the standard in any one year to be defined by the examiner.
4. That this Committee favours the institution of Honours courses in mathematics for the Bachelor's degree running parallel to the ordinary courses.

I add a few general comments. Even the 'good pass' referred to in clause 3 is hardly sufficient for university work. There is obvious need for a much greater measure of co-operation between the schools and the university colleges both in drawing up the examination syllabus and also in deciding on the results of the examinations. This might be achieved through a really active Board of Moderators. I understand that the new prescriptions for University Scholarship mathematics, drawn up in 1948, fit the Entrance prescription much better. It may be noted, too, that in the recent report of the Consultative Committee on the Training of Engineers[24] it is definitely recommended that two years be taken in the sixth form before the engineering intermediate course is begun.

Too many who study mathematics to sixth form standard abandon the subject at the university. For this, the university itself must take much of the blame. At Auckland, for instance, the Liaison Officer's figures[25] reveal that on the average only 30 per cent of those who enrol are successful in passing

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[24] *Education, Training, and Supply of Professional Engineers in New Zealand,* 31. *A. to J.* H-39 1949.

[25] for 1944-8.

Stage I, and those who enrol in this subject have already been carefully winnowed! Corresponding percentages in other subjects are as follows: geology 79, psychology 73, German 68, education 65, political science 60, geography 52, physics 50, chemistry 50, French 48, botany 48, English 47, history 46, applied mathematics 45, Latin 42.

The Liaison Officer enumerates sixteen causes of failure, stressing above all others 'academic entrance qualifications not high enough'. This, however, is singularly inapplicable to mathematics entrants who are, relatively speaking, very well grounded. Nor, apparently, is the Auckland University College unique in this matter. From a lecturer in mathematics at another university college I have received the following comment:

> I quite agree that at present students who could profit from further mathematics are being frightened away from the subject. This is certainly true of one group of students in whom I am particularly interested, those following the biological sciences. And again, we find that very few of our students from the Arts side take mathematics. It would appear that the prevailing motive for taking the subject is a vocational one, and mathematics is not taking its proper place in the cultural development of the students.

I do not presume to suggest to the university colleges the standards they should set, and am anxious not to generalize too readily. It does appear, however, that in a laudable effort to uphold standards, one college at least is strenuously discouraging average students from pursuing the study of degree mathematics. The University is rightly concerned with specialists, but it surely owes also a duty to non-specialist mathematicians who wish to take the subject to Stage I only. Intending primary school teachers naturally avoid the subject, with professional consequences that I have already deplored. The staffing of secondary schools is seriously affected. It is all very well to claim that only an Honours graduate can teach mathematics. This is a counsel of perfection which, rigorously applied to all subjects, would impose on the schools a narrow specialization that is impossible and, it may be contended, undesirable in New Zealand.[26] A Stage I standard that would permit average sixth formers—who after all have reached the University's own Entrance Examination standard in their subjects—to face mathematics with reasonable prospects of success is surely desirable and reasonable. At present, the Liaison Officer's conclusion sounds, in face of his own statistics, strangely hopeful: 'Anyone with average intelligence, who has been adequately prepared, who appreciates the necessity of self-discipline, and who is not afraid of hard work is certain to succeed at the university'.

Again, it would appear advisable that intending university entrants should remain for two or possibly even three years in the sixth form. Many of them are very young when they pass the School Certificate Examination. In these days of general economic prosperity, where child allowances and boarding bursaries are so freely available, this is surely not asking overmuch of parents in general. A closer co-ordination between school and college methods might then be possible. There could be more freedom and individual work in Form VI, and perhaps more teaching in the university. The implications of this suggestion cannot be considered here. The limited staffing of smaller secondary schools and of district high schools would probably make it advisable to have their senior pupils transferred on boarding bursaries to the city schools, as in fact is now being done.

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[26] In 1949, 78 graduates from New Zealand university colleges entered the graduate training course for teachers. Probably a few more entered the teaching service without training. Of college entrants, 3 had a Master's Degree with Honours in Mathematics, and 4 had B.A. Stage III qualifications in Mathematics.

The study of mathematics should not be lightly discontinued at school. It is obvious that for all students who desire to follow courses in science, engineering, architecture, indeed even in the social sciences and accountancy, a basic knowledge of the subject is essential. Many pupils today, for reasons suggested above, are following secondary school courses that will later prove quite inadequate as a basis for their professional studies. Mathematics is a particularly difficult subject for a boy to take up in home study or night school, and it should therefore not be lightly omitted from a school course. Most if not all boys of more than average ability might well take it seriously—at least to School Certificate standard. By that time, it would probably be apparent to teachers whether it was advisable to call a halt or to travel further. Incidentally, the success of mathematics candidates in the University Entrance Scholarship Examination of late years has raised some protest from language and social studies teachers. In the five year period, 1942-6, the top three places have been gained as follows: 13 by mathematics and science students; 1 by a Latin and mathematics student, and 1 by a language candidate. Ten of the successful scholars took 'additional mathematics'. The marks gained are very high. In 1946, for example, the top candidate scored 1294, the second 1286, and the third 1259, out of 1400 marks for mathematics, additional mathematics, and applied mathematics. These are, of course, selected samples. The average marks of candidates in the various subjects are much alike. But in a competitive examination an abnormally wide scatter of marks in any subject does favour the abler candidates in that subject.

The non-mathematicians' case was presented, with imposing mathematical calculations, at the Annual Conference of the Secondary Schools' Association in 1949. The valid conclusions to be drawn are doubtful. In boys' schools at least—and boys figure very prominently in these selected marks—the present vocational bent is towards science, and most of the best young brains in the country are choosing that line. The dispute is a family one, outside my present purpose; but at least it indicates a cheerful prospect for mathematicians. (The report on the *Higher School Certificate* discussing 'The Machinery of Examination' pp. 27-40, considers also how standards of performance in two subjects can be equated. The influence of the subjective factor—the examiner's personality—is shown to exist in spite of any statistical system that may be adopted.)

§4 TECHNICAL SCHOOL MATHEMATICS

Oversea readers may be surprised to find that in these pages no specific reference has so far been made to the teaching of mathematics in technical high schools as such; but in fact these have been borne in mind throughout. It has to be remembered that pupils in these schools are required to study the same core of subjects as are the secondary school pupils. School Certificate, University Entrance, and Form VI mathematics are likewise imposed in the same form on both types of school. Indeed, in the majority of instances, the distinction of types is itself misleading. In the four main cities and a few other centres, so-called technical schools are distinct from grammar or high schools. Otherwise the typical post-primary school is a multi-purpose school, and its particular title is a matter rather of local fancy (not altogether guiltless of some reference to possible financial consequences in the choice). Nor can it be assumed, where separate institutions co-exist, that the ablest pupils are to be found in the academic school. Times change.

One clear difference does exist between the larger technical schools and their grammar school comrades. The engineering departments in the technical colleges of the larger cities are efficient, business-like, well-equipped departments, vital to the college, and no mere ancillaries to academic studies. Well-trained technical staffs carry out the instruction, and conscious effort is definitely and persistently directed towards keeping in touch with industry. This essentially practical department obviously discourages a merely theoretical approach to mathematics. The subject becomes the handmaiden of science, rather than its queen.

The peculiar features and advantages of mathematics teaching in such colleges merit some elaboration, for they may point the way to more effective treatment in other types of school. In the first place, the pupils commonly have a sense of meaning and purpose in their studies, and this is strengthened by the varied types of practical application that abound. Where the principal and staff have faith in its power, mathematics can be a stabilizing and vitalizing force in all the courses. Pupils who are not interested in the School Certificate Examination as such may replace it by relevant trade or commercial examinations. The urgency with which the budding tradesman requires the mathematical equipment to grapple successfully with the problems of electricity, heat engines, and mechanics; the evident advantages to the commercial aspirant in her book-keeping of good tabulating with neat figures, of accuracy in particular fields of arithmetic, and of diagramatic representation; the obvious bearing of budgeting problems, mensuration and graphs on the life of the domestic student (and indeed the fuller mathematics required by those who would graduate B.H.Sc.)—all these form powerful motivating forces encouraging a live response from pupils and a keen effort from teachers.

A second virtue follows easily. Not only do the practical tasks undertaken plainly require mathematical skill in calculating volumes, weights, resistances, energy, and so on, but they drive home the necessity for care and accuracy. A problem wrongly worked in an academic form may incur the censure of the teacher; a machine wrongly used, a measurement carelessly made, a miscalculation of stress and strain, may wreck a whole undertaking. If the undertaking has significance for the pupil (and teachers see that it has) the resultant failure carries its own lesson. Mathematics as a prelude to action is seen as an obvious need. The same boys who object to aimless exercises involving $\frac{abc}{d}$ will toil willingly and enjoy substituting values in a formula and evaluating $H = \frac{PLAN}{33000}$. One is algebra, the other engineering!

Moreover, the influence of teachers whose main concern is with practical applications of mathematics is felt strongly throughout the college. Co-operative planning by a group of mathematics teachers, some drawn from the work bench, some from the office, some from the university or training college, can become an enjoyable search to determine content, to decide on the order of topics, and to provide reasons for every step. From such team work each member will emerge the richer. The practical instructor learns how to present the work in the classroom and gives in return reality and purpose to the pedagogy of the trained teacher. Again, the enlistment of support from teachers of allied subjects ensures that every effort is made to prepare pupils satisfactorily and that the basic mathematical methods agreed on are followed throughout the school. If barriers between classrooms are broken down, uniformity of presentation of similar processes will prevent confusion in the pupils' minds and indeed will strengthen their grasp of principles.

Further, the core studies and activities are necessarily interpreted quite differently in the various courses, so that the danger of blind acceptance of a prescribed and general syllabus of work is avoided. Boys, interested in building and engineering, must concentrate on algebra, trigonometry, practical geometry, mensuration, logarithms; girls, whether in domestic, art or commercial courses, have common interests in cooking, dressmaking, and accounts, and their mathematical training should naturally reflect these interests. There may be little apparent similarity between trade and commercial mathematics courses, but the core syllabus is wide enough to embrace both; and courses can be devised to satisfy both core requirements and the special needs of individual students.

The teacher who bears in mind the later vocational needs of his pupils will scrutinize carefully each topic in his syllabus to see if its inclusion is justifiable. By the removal of dead wood and the use of enlightened teaching methods, rapid progress can be made and the essential mathematical skills can be secured in good time.

This note of urgency in respect to the mastery of fundamental skills is peculiarly strong in technical schools. These schools are by their very nature driven to realize the value of co-ordinating the work of all subjects of the curriculum; and the insistent demand for early manipulative skill in mathematics provides an incentive for experimentation to discover how best to arrange the order of topics and to cover the ground rapidly yet effectively. The virtues of an unorthodox approach such as the rapid treatment of algebra through formulae to equations with the basic processes explained and drilled when required; the value to be derived from an early working knowledge of graphs, trigonometry, logarithms, practical geometry, unencumbered by the perplexing theory that can readily be assimilated by the more mature student; the desirability of common-sense mathematical training for girls including really live arithmetic, graphs, some facility with symbols, some geometrical appreciations—all these are now recognized in practice in technical schools. Their acceptance by the more academic schools, with necessary modifications, can do nothing but good. Finally, the experience of technical colleges has demonstrably justified our faith in the ability of non-academic pupils to master valuable mathematical techniques. Previously condemned to further study in arithmetic often quite unsuited to their needs, weak pupils have found new hope in the appeal of graphs and relatively easy manipulations in algebra, and enthusiastically respond to remedial work in arithmetic based on sound diagnostic testing. The variety and breadth of teaching practice in a technical school may make the work arduous—but it is never dull, often encouraging, and generally satisfying.

It is therefore not surprising that some of the most successful teaching of mathematics is to be found in these departments or schools.

In multi-purpose schools, where academic classes spend some time in workshops, or at least mix with others who take the 'practical' courses, comparable success may be achieved. Close co-operation between theoretical work in the classroom, and the problems to be met in the workshop is an obvious desideratum. Where, as commonly happens, theory and practice are handled by different teachers (academic and technical) this co-operation should be definitely sought. It appears that occasionally the two departments have no dealings with each other. The academic theorist may despise the tradesman, the skilled craftsman barely conceals his contempt for the helpless theorist. It is a pity. Some day, perhaps, mutual respect and trust will be generated. Meantime, as a repentant theorist, I would urge my fellow academics to seek the company of their technical colleagues. It is easier for them to visit the workshop than for the technician to leave his machines. How long will foolish pride and groundless prejudice hold us apart?

IV

METHOD IN TEACHING

THE TERM METHOD MAY BE LEGITIMATELY APPLIED TO TWO very different though closely associated ideas. In a general sense, it may signify the broad approach to a subject—logical or psychological, theoretical or practical, formal or informal, discipline-poled or interest-poled, and the like. We may have a project method or a play-way, an expository or a heuristic presentation, an integrated course or a set of independent units, a popular or a philosophical approach, and so on. The choice of method in this sense of the term is of the highest importance, as it sets the spirit and substance of the whole course. The Department's syllabus does in fact strongly encourage a modern or realistic method, the justification of which depends upon psychological considerations, and on philosophical and psychological values and aims. Some brief indication of the nature of this justification is attempted here, but obviously it can be no more than an indication.

On the other hand, given these aims, this content, this general approach, we have still the more intimate problems for the teacher—the handling of specific topics, and of preparing, presenting, developing, checking and applying new ideas. This is classroom technique. The distinction is commonly noted in teachers' training colleges by the double-barrelled term, 'Principles and Practice of Teaching'. With the recklessness of a professional lecturer ('Those who can't teach, teach others to teach', as my students unkindly quote) I add some specific suggestions with regard to classroom procedure. These must be taken as suggestive and illustrative only, and written for the non-mathematician who must try to interpret the syllabus. This is not a manual of method; I have no desire to teach my betters their job.

§1 CLASSROOM PROCEDURE

The traditional approach is essentially formal, logical, systematic. It recognizes several distinct types of mathematics, and pursues these concurrently but in isolation. Arithmetic, indeed, used to be regarded as a primary school subject; and in the good old days when children leaving primary school really could do arithmetic, a period a week at Workman, just to keep one's hand in, was all that the secondary school need provide. Algebra worked steadily through mechanical operations, removed or added brackets, solved simple and simultaneous equations, turned aside for a section on graphs, factorized various prepared examples of possible and impossible expressions, moved on to quadratics (with problems), and concluded with harder examples in factorizing, problems, and simultaneous equations. The underlying theory was a logical one: if children were to use mathematical methods in solving problems, they must first master thoroughly and in isolation the basic mechanical operations. With the turn of the century, geometry in New Zealand broke down the logical austerity of Euclid, and the *pons asinorum* was no more; but the subject still consisted essentially of a series of propositions to be mastered and reproduced, with exercises thereon which might or might not be worked.

It is important that we should state explicitly the aims and assumptions associated with this method, and I tabulate what appear to be the more important as follows:

(1) Mathematics is regarded as essentially a logical discipline. It is admitted, for example, that all the algebra and geometry mastered by a pupil leaving from a fourth or fifth form has little direct practical value or application. It is not anticipated that he will meet with quadratic equations in his everyday life, or be particularly concerned with medians of a triangle. It is even admitted that he will soon have but the faintest remembrance of what it was all about. But, it is claimed, he will have learned more valuable lessons:

(a) He will have learned to be accurate. The working of countless examples in arithmetic and algebra and the regular formal statement of geometrical proofs will have worked into his very system the ideal of exactness, of precision.

(b) He will have learned to work logically and systematically, building up knowledge and skill step by step so far as he goes. This is training that he will get nowhere else, though perhaps Latin and science may have somewhat similar virtues. And, to his soul's good, he will have done this without the spur of interest or utility. For the disciplinary view is commonly, but not inevitably, associated with the idea that it is 'good for' a pupil to have to study what he does not like. The inherent contradiction in this attitude is seen when we realize that on this assumption those who *like* mathematics *ipso facto* lose much of the value of the subject.

(c) Particularly, he will have been trained to recognize what Spearman has called 'the relation of evidence'. He will have learned, that is, to recognize and make explicit what is implicit in his data. Given that two different angles are subtended at the circumference by the same arc, he will realize that they must be equal, and that each must be half a corresponding angle at the centre. This is pure logic applied to shapes and figures, and has much the same virtues as a treatise on formal deductive logic. Polya claims that 'a student who fails to become acquainted with geometric proofs has missed the best and simplest examples of true evidence, and the best opportunity to acquire the idea of strict reasoning.'[1]

(d) Even more important, he will learn the art of 'postulational thinking'. This '*if-then*' kind of thinking which makes the pupil conscious of his premises, his assumptions, his postulates, has grown to be of the greatest importance not only in the physical and biological but also in the modern social studies. Research workers in education, sociology, economics, and almost every field employ it widely.

In a word, it is claimed that the discipline of formal mathematics teaches a pupil to think clearly and accurately, to recognize valid evidence, to avoid common logical errors such as *petitio principii*, *circulus in definiendo*, *non sequitur*, and to proceed systematically from known to unknown. It obviously implies also acceptance of the doctrine of transfer of training.

(2) This attitude is closely associated with the view that mathematics is a pure science—is, indeed, the most accurate of sciences, the queen of them all. Sir James Jeans, it will be remembered, claims the Creator as a pure mathematician.

In a well known passage in his *Algebra*, Nunn distinguishes two aspects of mathematics thus: 'Mathematical truths always have two sides or aspects. With the one, they face and have contact with the world of outer realities lying in time and space. With the other, they face and have relations with one another.' He then proceeds to emphasize the complementary nature of these methods, which act and react on each other. It is on the second aspect of mathematical truth that the traditional approach lays stress—unfortunately, in the opinion of the Spens Report.[2]

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[1] G. Polya, *How to Solve It: A New Aspect of Mathematical Method*, 189. Princeton 1945.

(3) This traditional outlook had led naturally to those formal methods that have already been noted, and to emphasis on deductive thought to the neglect of inductive.

(4) Traditional ways are supported by several considerations, all with some force. Most elderly teachers were themselves trained successfully along these lines, both in school and in university, and are therefore biased in its favour. Indeed, it is probably true to say that specialist teachers of mathematics tend to be precisely those people to whom the beauty and logic of abstract mathematics most appeal. There is abundant scope in industry and applied science for those who prefer the practical side.

Moreover, the method is easy to handle. There is a definiteness about the work that appeals. Exercises in textbooks are numerous and well graded, and the ground to be covered is clearly marked out and signposted. Everything is orderly, like a Dutch garden. Neat, clear, tidy, methodical—what more can man ask? Besides, the method works. In the limited field of study, there is a certainty and a sense of mastery that are not found elsewhere. Examination prescriptions can be met with confidence, and teaching technique is simple. Group and individual work can be successfully combined; and the experienced teacher, knowing exactly what is ahead, can anticipate later difficulties and evolve methods of explanation and approach to meet his personal needs. Altogether a very satisfactory state of affairs. Modern teaching methods make much heavier demands on the teacher, and in the hands of an incompetent may prove disastrous. The old method may be limited, but within those narrow limits it is safe. Its definiteness also appeals strongly to those teachers who are not really specialists in the subject—and under present conditions there must be many of these. They can always fall back on the book. Finally, emphasis on formal and mechanical work suits examination conditions. The examiner can set and mark his papers with more ease and confidence and the teacher can make sure of bookwork—factual knowledge can be memorized (even if power cannot be crammed).

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[2] *Secondary Education with Special Reference to Grammar Schools and Technical High Schools*, 236.

These assumptions may be partly unconscious, and some may even be disavowed when they are thus stated explicitly. They commonly manifest themselves in such forms as the following: a call for thoroughness and accuracy as against alleged superficiality; an exaltation of the cultural or the logical, and a rejection of the merely useful and practical; a general rejection of modern psychological methods as pandering to mediocrity and the desire to avoid effort; a plea that teaching and examination conditions preclude the use of modern methods (which are thought to require more time), impose uniformity throughout classes and schools, and favour the use of good standard textbooks; a claim that traditional methods are sound and successful; an argument that, whatever the examination result, the experience of 'rigid logical proof' is a discipline that transcends all merely practical gains; a demand for convincing proof that new methods are superior to the old ones they would replace; an appeal to the university colleges, which (themselves essentially interested in pure mathematics) call for a 'sound' foundation on which to build, and tend to blame the new methods for the inaccuracy so commonly found in Entrance Examination papers.

The modernist claims, on the other hand, that this logical approach is practically ineffective, pedagogically unsound, and even mathematically misleading. The practical failure of the old approach is, of course, relative only, and not absolute. In particular, the method has met with some success with pupils of high intelligence and a natural flair for mathematical abstractions. Candidates who select the subject for the University Scholarship Examination may reach a very high standard indeed.

In his well known address (1901) to the British Association Perry criticized the prevailing method of teaching geometry in these words: 'So now we teach all boys what is called mathematical philosophy, that we may catch in our net the one demigod, the one pure mathematician, and we do our best to ruin all the others. It is nature's way with fishes: 10,000 herrings spawned for one survivor: 10,000 Tom's, Dick's, and Harry's mentally destroyed for the sake of producing one man fit to be a mathematical master of a second-rate public school; ten million destroyed for the sake of producing one great mathematician.'

Apart, then, from the gifted few, how shall we evaluate the result of our mathematics teaching? It is obvious that all the mathematical knowledge and skill that an average pupil can gain in three or four years at a secondary school must be small. A similar criticism holds, of course, of every school subject. How can we gauge the value of that little? There is no agreed criterion, but the following criteria, though partial are relevant.

First, the popularity of the subject with those who have studied it. This, to the strict disciplinarian, is a meaningless test; discipline, he would affirm, is rarely popular, and there is a virtue in a subject that demands intense effort without the spur of curiosity, pleasure, amusement, or immediate practical use. But let us hold over this doctrine till we begin our psychologizing. Two important facts must be borne in mind here. Pupils who leave school with no liking for a subject—let alone a dislike for it—will not pursue it in after years, so that any value its school study has had for them must be very intense, pervasive, and tenacious of life if it is to modify their general thought and conduct in adulthood. Furthermore, dislike of a subject is very largely a reflection of failure to understand or make progress in it. We like subjects that we can do, and dislike (and disparage) those we cannot. The urge to uphold our self-respect is fundamental.

Now there is general agreement among writers the world over that mathematics is an unpopular subject. *The Content of Education* introduces its chapter on mathematics with these arresting words: 'Is the unpopularity of mathematics inevitable? For unpopular it undoubtedly is . . . . The attitude of those who dislike mathematics is not merely one of apathy, but intense hostility. And among those who do like it are to be found emotions approaching fervour and even ecstasy. These are strange and intriguing facts.' Strange indeed!

As to systematic thinking, postulational or practical, it is the common testimony of teachers of weaker forms that pupils must be steadily crammed in bookwork if they are to pass an examination. The atmosphere of hopeless defeatism that fills the classroom depresses the visitor. He senses that the teacher himself sees no purpose in his teaching; the class expects nothing, and does nothing. This lack of class effort, by the way, rather disposes of the claim that pupils at least learn to work hard. It is surprising how very little a boy will try for himself, how ready he is to protest 'we haven't had this kind', how satisfied to plod along mechanically.

Furthermore, mathematical situations and problems of the traditional textbook type are completely remote from the experience of everyday life. There, a man must often work with confused or incomplete data, and find his own guiding principles. The systematic formalism of a textbook, like that of logic, over-simplifies the problems of life. It is a reasoning device to help the recognition of the relationship *of* evidence as distinct from the cognizing of relations *by* evidence—a far more important power.[3] This artificial exercise of reason is like a special exercise in a gymnasium, in its relation to the affairs of normal life. If the presentation were made inductive and experimental as well as deductive, if data were sometimes superfluous, irrelevant, or insufficient, thus compelling selection to be made or further data to be gathered, if it were not always limited to proving one conclusion (and that a given one) at one time[4] then some approach to the demands of everyday life would have been made, and a more reasonable hope of transfer of method and skill would be established. A functional approach makes all this possible.

Now, unless we can establish that our academic teaching of mathematics has in fact given our pupils a sound training in reasoning, we can hardly justify it at all. The practical, everyday knowledge gained in the traditional four-year course represents little of general value in life. The average pupil under the old traditions left school after four years' work with no conception of the real meaning of mathematics, no skill in handling the little he had learned, and no desire to have any more truck with the subject. In 1900, Smith[5] wrote: 'As a valuable training in logic, as a stimulus to mathematical study, and as a foundation for future research, the study of Euclid as undertaken in England was not a success.' And Minchin, an English professor of mathematics, wrote at the same time: 'For myself I confess that, to the best of my belief, I had been through the six books of Euclid without really understanding the meaning of an angle.'

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[3] See C. Spearman, *The Nature of Intelligence and The Principles of Cognition*, Chapter V. London 1923.

[4] cf. van Zyl, op. cit., 126.

The pedagogic unsoundness of the traditional approach is readily demonstrable to anyone familiar with modern psychology; but as the traditional mathematics teacher is rarely found in such doubtful company, the task of convincing him is rather formidable. For the virtues that he exalts are very real and fundamentally important. Only if we can reassure him that they are safeguarded in the new approach, may we hope to win his confidence. The Spens Report claims to have said 'sufficient to show how the amount of time now given to mathematics may be reduced without loss to the value of the subject.'[6] In a final comment on its suggestions for improved teaching, it says that the course in mathematics 'will ignore many topics which are still included in textbooks, but have little value for any except the professional mathematician, and not always for him. But trigonometry and the calculus will no longer be sealed books for the majority of pupils in the schools.'[7] The

[5] D. E. Smith, *The Teaching of Elementary Mathematics*, 94. Ginn, Boston 1900.

[6] Spens Report, 242. [7] ibid., 241.

purpose of the new approach is to do more work more effectively in less time, by adopting psychologically sound methods. The formal, logical approach to a secondary school subject is educationally unsound because it ignores the natural limitations of childhood. Education cannot be imposed on a passive mind, but must meet the needs and be attuned to the powers and interests of the recipient. Misconceptions about the nature of interest, and confusion of interest with amusement or pleasure are still common. The world may be unpleasant (and today it promises to be very unpleasant indeed) but it ceases to be interesting to us only when we die or completely lose consciousness.

The modern introduction to mathematics is essentially practical. It presents the subject as a tool to be used in solving problems that have faced, still face, and will continue to face man. At every step in its progress it answers that most persistent and embarrassing question, 'Why should we learn this?' And if it does not reveal the final meaning of mathematics, it at least carries the student steadily forward to a more adequate conception of its meaning.

The Content of Education (Chapter VIII) finds the root trouble in mathematics teaching in 'the policy of splendid isolation pursued by so many mathematicians', and continues: 'They try to develop mathematics as a state without a foreign policy. They ignore its active commerce with neighbouring states and seek to develop their own soul unsullied. Such aloofness inevitably begets unpopularity. And this policy is the ruination of mathematics as an educational subject.'

The practical approach, therefore, is psychologically acceptable. It gives meaning to the subject, emphasizes its unity in diversity, provides an incentive to study, gains clearness and reality by basing its abstractions on concrete examples, avoids the use of what to average children is meaningless phraseology, mere incantation, and endless mechanical manipulation, suits the rigour of its demands to the developing intellect, and in American parlance, 'gets going'. 'Teaching the mechanical performance of routine mathematical operations and nothing else is well under the level of the cookbook because kitchen recipes do leave something to the imagination and judgment of the cook, but mathematical recipes do not.'[8]

These faults or weaknesses are perhaps enough to be going on with; but there is a still more serious aspect of criticism. Little hint is given in the formal course either of the real nature of mathematics or of its methods, and by suggestion or implication many false concepts are engendered. I was recently coaching a secondary schoolboy when we came to a geometrical exercise which temporarily baffled us. The pupil was suddenly struck with a great idea, 'What if the class teacher could not work out a problem himself?' Our preoccupation with established theorems, and with set exercises familiar to the teacher leads to the impression that mathematics is a subject of established limits and prescribed methods. That mathematics ever had a first dawn is barely suggested in the old textbooks, which present a complete and orderly course, all logically arranged, and advancing along the separate fronts of arithmetic, algebra, geometry, and trigonometry to their predestined goal. The young scientist might, indeed, hear of Archimedes starting from his bath and rushing with his triumphant 'Eureka' to announce his discovery, but we who began in the days of Euclid's reign learned nothing of the old gentleman himself, still less

[8] Polya, op. cit., 159.

of his ideas in their growth. In retrospect, indeed, I doubt if we ever thought of him as a man. He was a voice, the alpha and omega of geometrical reasoning, the unchanging and the unchangeable. That his was not the first or the only system never approached the threshold of our consciousness. Today, we realize that mathematics is 'a furiously expanding subject', in keeping with the revolutionary ideas in all branches of science, physical, social, and moral. 'The nineteenth century alone contributed about five times as much . . . as had all preceding history.'

Our modern textbooks move somewhat sedately along freer lines, and there is still room for more joyous abandon, more daring adventures. Such books as Hogben's *Mathematics for the Million*, Sawyer's *Mathematician's Delight*, Polya's *How to Solve It*, McCormack's *Mathematics for Modern Life*, commend themselves with their realistic attitude, their reference to everyday life, their encouragement of original thinking, their illustration of historical development. Mathematics in these circumstances becomes meaningful and even possible for average minds. (I do not for a moment suggest these as textbooks. Only the last-named makes pretension to textbook status: the others stress general attitudes that are lacking in most textbooks.)

It should be remembered, too, that the whole background of thought today is new and changing. In 1826, the School Board of Lancaster, Ohio, could write the following incredible words: 'You are welcome to use the schoolhouse to debate all proper questions in, but such things as railroads are impossibilities and rank infidelity. There is nothing in the Word of God about them. If God designed that His intelligent creatures should travel at the frightful speed of 15 miles an hour by steam, He would have clearly foretold it through His holy prophets. It is a device to lead immortal souls down to Hell'. Today the secondary school boy is quite familiar with concepts of the infinitely great and infinitely small in time and space, lives in a world of speed and change, of forces and resistances, of strains and stresses, of changing velocities and accelerations. Modern mathematics leads directly to the calculus and to analytical geometry. The leisurely and abstract logic of a former age, lacking in 'that sense of the purposefulness of mathematics without which it may so easily become little more than a dead language'[9] holds little of human value and meaning for modern children.

Finally though a school textbook must necessarily fall far short of the completeness of advanced mathematics, it should lead naturally towards the great principles that form the essential characteristics of the subject. These are given by Whitehead as the notions of the variable, of form, and of complete generality. In his own words 'the growth of science is not primarily in bulk, but in ideas; and the more ideas grow, the fewer are the deductions which it is worth while to write down. Unfortunately mathematics is always encumbered in textbooks by numberless subsidiary propositions, whose importance has been lost by their absorption into the role of particular cases of more general truths.'[10] It is not impossible that these features should characterize the treatment of school mathematics. The Spens Report, in complete harmony with Whitehead, says:

> There are three ideas which claim precedence, not only because they arise naturally out of attempts to apply mathematical ideas to practical problems, but also because they

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[9] Spens Report, 238.
[10] A. N. Whitehead, *Introduction to Mathematics*, 115. London 1925.

are *the necessary preliminary to other ideas that will be developed later.*[11] These are: (a) the use of algebra as a symbolic language . . . or in other words the use of the formula and the manipulation of the formula . . .;
(b) the classification of spatial ideas which is the theme of geometry . . . . Two other ideas should be developed early, that of symmetry of form and of sameness of shape or similarity;
(c) the use of the graph as possessing something of the nature of a pictorial formula and as an important instrument of analysis and generalization of which much use will be made later.[12]

The modern approach, therefore, differs fundamentally from the old. Not only does it depend, as we have seen, on a new conception of the essential nature and purpose of school mathematics, but it substitutes a psychological for a logical basis of learning and teaching. This means simply that we recognize the nature and limitations of children's interest and intellectual powers, and present our material accordingly. In particular, we realize that a normal child is not a reasoning machine, finds difficulty with abstractions, is confused by formalism and symbolism, has a very limited experience of life, and is not driven by a passion for pure knowledge. We realize, too, that the secondary school child develops while at school, so that the sixth former should be capable of sustained abstract reasoning with mathematical formulae, and have developed a genuine interest in mathematical thought, though in the third form these were beyond him. This recognition that differing abilities and developing skills and interests should determine method is fundamental. Speaking to graduate students on teaching methods, a secondary school inspector recently insisted that sixth formers must be treated as equals, that they must not be spoon-fed, that they were 'more sensible than undergraduates', and that their examination tests should be worthy of their abilities and their sense of personal dignity.

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[11] The italics are mine.

[12] Spens Report, 240.

From the teaching standpoint, the main objections to the new approach are these. First, it throws much more responsibility on the teacher, who must be alive and alert. He can no longer merely keep his flock together and moving steadily along the recognized stock route. Again, this adapting of material and methods to the individual differences of children is difficult, and may lead to error and confusion. The specialist teacher may be weak in personal relationships, and indeed is not infrequently an indifferent teacher of beginners. Further there is the real danger that accuracy and thoroughness may be sacrificed for interest and superficial progress, or that the work done may reflect too faithfully the faults and idiosyncrasies of the teacher, as well as his strengths and enthusiasms. The new method certainly requires a teacher who knows and is genuinely interested in his subject, and in addition understands children. Such teachers, in New Zealand at any rate, are not over-plentiful, and we cannot build on the assumption of their presence in all schools and with all classes. Finally, it may be claimed that much alleged psychology is unsound and misleading; as understood by many teachers it most assuredly is.

§2 SOME MAXIMS OF METHOD

All techniques should be based on insight.[13]

It should first be emphasized that there is, and can be, no such thing as 'the right' method of teaching anything.

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[13] National Council of the Teachers of Mathematics, *Fifteenth Yearbook: The Place of Mathematics in Secondary Education*, 8. New York 1940.

Method is relative to circumstances—to the class, the teacher, the ultimate aim, the aim of the moment, classroom conditions. Some of these change from day to day; all suffer a long-term change. So the teacher who begins his career with methods suited to his immediate needs commits professional suicide when he accepts these as permanently and absolutely the best. And, of course, the teacher who is constantly changing his methods from mere love of change or the desire to be up-to-date errs as seriously in the opposite direction. The terms 'old' and 'new' as applied to methods are in themselves devoid of all judgment value.

Broad principles, however, are vitally important. Stated in terms of maxims, they may incur contempt as being too obvious and childish for words. Yet truisms have their place in life. For one thing, they happen generally to be true. For another, they are commonly (to our incalculable loss) ignored in practice. I am here not so much concerned with textbook maxims, such as the well known sextet: proceed from the known to the unknown, from the concrete to the abstract, from the simple to the complex, from analysis to synthesis, from empirical to rational, from indefinite to definite. These, as Welton (with his typical perversity) points out, must be interpreted in practice, and may lead us astray as often as they lead us aright. A fifty-fifty chance of success or failure is not very encouraging.

With considerable diffidence, but impelled by a sense of their importance, I suggest the following. The wording and the presentation are my own, but the ideas themselves are mere crystallizations of views accepted and expounded by educationists and teachers throughout the English-speaking world. They are not exhaustive, but they may serve their turn.

First, we have the dictum of Sir John Adams: 'A thing is not given until it is taken: a lesson is not taught until it is learned'. The soundest exposition fails if it is not accepted. This is frequently overlooked. We teachers too readily rationalize our methods and assume the success of our efforts, steadily believing that our English teaching produces a love of literature, our French, an appreciation of culture, our mathematics, a love of truth, accuracy and logical presentation, our science, a passion for truth and scientific method, our social studies, good citizenship. Those who deal with the products of our teaching are justly sceptical; at times one even hears charges of failure.

Secondly, we have the test of learning itself—a lesson is not learned until it has become part and parcel of the pupil's mental attitudes or personality. The verbal statement of a geometrical proposition may be correctly reproduced by a class that has learned nothing from its logic, and cannot use it in simple application. The error of verbalism is rife in the world today. We play with abstract terms—truth, justice, accuracy, culture, democracy, progress, Christianity, religion and the like, but in practice these ideas have perhaps never been so generally ineffective in the world as they are today.

Thirdly, our teaching must ensure some measure of completeness at every stage of school life, at the same time preserving the unity of the long-term whole. The fact that pupils in a secondary school may leave at any time complicates all schemes of study. The old logical method had little meaning till a considerable amount of work had been done, for three or four years. It prepared for a future that was rarely realized. Obviously, no method will make a one-year course as good as a two- or three-year course, but it may render that course valuable in itself. A third-form child, for instance, who learns to read and interpret simple graphs has learned some mathematics of real everyday value, even if his post-primary schooling goes no further than the first year. This leads simply and readily to the further requirement of interest, which has been so recklessly equated with play, amusement, constant activity, avoidance of sustained effort. It should be understood that the appeal to interest is a call for more intense study, not less. 'The new teaching,' writes Sir John Adams, 'does not seek to get rid of drudgery, but to make it tolerable by giving it a meaning and showing its relation to the whole learning process in school and to the whole process of living in the world.' To secure that interest, we must recognize and meet the present needs of our pupils. We must accept the psychological as opposed to the purely logical method of approach. The mathematician, Whitehead, devotes an address ('The Rhythm of Education') to this principle, which he dubs a truism that educational practice has insufficiently recognized. It means simply that we recognize the limitations of childhood, and seek to conquer nature by obeying her.

As a corollary to this, I would stress the need of thorough mastery of essential groundwork. This is particularly important in a systematic subject like mathematics or Latin. Uncertainty in mechanical operations, in the meaning of symbols, in the grasp of fundamental principles not only leads to error and ultimate failure but also paralyses effort, induces a feeling of utter helplessness, with consequent fear and hatred. Thorough mastery of the work to date gives to the pupil a feeling of confidence and a sense of power that provides a strong incentive for further effort. The psychologist, Adler, made this desire for power the basic factor in human conduct—it certainly is one of our fundamental drives.

Along with this consideration, and frequently tending to confuse it, is the principle of anticipation. There are times when some explanation of a difficulty must be given, and the correct one would be unintelligible. In such a situation, we do the best we can. But, knowing what is ahead of our pupils, we so direct our course that they never suddenly come to what, borrowing geological language, we may term 'a fault'— a complete break with the past. Suggestions of the calculus, of solid geometry, of imaginary numbers and incommensurables, will be foreshadowed even in junior forms so that later theory becomes merely an extension and intension of rough ideas already adumbrated, and receptive mental attitudes have already been developed.

Two more specific maxims:

We must teach with the aim of securing an *understanding* of our subject so far as we go, and a desire to continue the study further. The two 17-year-old girls (just matriculated) who rejoiced: 'Think of it! No more maths! Never to open another geometry book!' exemplify the failure of unenlightened study.

We must wait for nature, and not try to force the pace. It is so easy, when criticizing the efforts of, say, Certificate candidates, to fall into the absurd error of thinking of these as the result of three or four years' work at the sixteen-year-old level. A third-year candidate aged 16 was 14 in his first year, and 15 in his second. His examination age may lead us to overlook the simple fact that both the mentality and the character of the boy have been changing throughout. In our haste for results we are too easily impatient of the immaturity of our pupils. Above all, the first year of secondary school mathematics must be unhurried. The work must suit the ability of the class, and proceed at its normal pace. A weak foundation is a handicap in any subject; in mathematics, it is simply disastrous. We must be sure of our fundamental training. If we must hurry, in deference to external demands, let us do so in the last year, when it is to be hoped that our pupils will be sufficiently well grounded to meet the increased pressure. This will normally occur. The temptation to mark off the syllabus into three or four roughly equal sections, one for each year, is to be avoided at all cost. Similarly, the grim determination to 'do' the various topics, to cover the ground somehow, is to be condemned.

Finally, and summing up the whole doctrine of modern teaching, the method adopted should at all stages make the whole procedure meaningful and purposeful: the wood must first be seen as a whole, not the individual and apparently separate trees. An increasing and developing meaning and purpose will give to the study that interest and appeal which makes for continued success.

§3 THIRD FORM METHOD

Well begun is half done, quoth Hendyng.

These maxims may well be illustrated by a more specific reference to classroom methods in the third form. It is here that the seeds of failure are too commonly sown—confusion, uncertainty, discouragement, fear, hatred. It is here that real teaching is necessary; it is here that the battle is often lost before it is properly begun. At this stage, the pupil needs above all interest, encouragement, success. His teacher must have patience and sympathy, restrained enthusiasm and system in his approach. At all costs he must inspire self-confidence in his pupils. Scolding them for weakness in fundamental arithmetic, failing to adapt his presentation to the needs and powers of his class, overstressing formalism and driving relentlessly on—these are the common faults, it would appear, of poor third-form teachers.

The first need of the third-form teacher is a good textbook. There are subjects where the textbook matters relatively little, but mathematics is not one of these. A supply of useful material, properly graded, must be available to the class, for mathematics is essentially a matter of individual doing, of practice. But more important still, the book must have the right approach to the subject, and the right type of example. The subject is a difficult one, and disagreement between textbook and teacher is confusing and disturbing to the pupil. No matter how well the teacher presents the matter the individual pupil must constantly refer back to the text for clarification. Where the teacher jumps about in the book, the logical development of the text is destroyed, examples are less suited to the teacher's approach, and the pupil is conscious of divided authority.

There is, so far as I know, no satisfactory textbook in arithmetic for use in New Zealand, and the need is very urgent. Mathematics masters are agreed on this. Arithmetic is necessarily a somewhat more loosely organized subject than algebra or geometry, and it readily becomes a meaningless jumble of isolated topics. Reference has already been made to the present Form II Arithmetic book. One of its most serious limitations is that it is not linked with Form III and IV books designed to develop its treatment; indeed it is rather a finishing-off text for primary school leavers than a preparation for secondary school work. When this new textbook is written, extreme care must be taken to see that it avoids the weaknesses we have considered, and that it embodies the new approach to arithmetic. It will contain some elementary work in graphs and statistics, the symbolism of algebra, and the study of geometrical form, in addition to the more purely arithmetical matter. Because for other pupils it must be used also with texts on algebra and geometry, treatment of these topics must be unified. To treat graphs from three different standpoints (in arithmetic, algebra, geometry respectively) is obvious nonsense. It is possible that third forms would be better served with a single textbook on mathematics, where all three aspects of the subject are naturally integrated.

It would be reckless of me to forecast the form of such a textbook. I have received detailed suggestions from various experienced teachers, whose views should be long and carefully considered by the writers of the new book, but cannot find room for them here. Several attempts have already been made, but not very successfully. It is apparently a most difficult problem to handle for a course of two or three years' duration. Some American courses leave out arithmetic—surely an unsatisfactory solution.

The general approach to geometry is embodied in several accepted textbooks, of which Siddons and Snell's *Introduction to Geometry* or Durell's *Geometry Stage A* may be taken as typical. Teachers who have trouble in grasping the plan of these texts would do well to read many times the Second Report on the *Teaching of Geometry* referred to so frequently above. It is essential to grasp the idea of three different levels or stages in the handling of geometry, and not to introduce the later stages too soon. The new attitude towards geometrical proof is also of fundamental importance. Geometry is no longer a procession from one abstract truth to another. On the perceptual or experimental level, the child first realizes certain geometrical truths. The proof of these follows such realization, and does not precede it. He discovers by actual measurements, for instance, that the three angles of any triangle contain altogether a total of 180 degrees, within the limits of practical error. He is led to wonder, 'Is this always true?—If it is, why must it be so?' A proof that meets this wonder is successful. A third important feature is the grouping of propositions, so that a key proposition gives immediate acceptance of various associated propositions. The propositions dealing with angles in a circle form such a group, and illustrate very well the principle of grouping. Once it has been demonstrated that the angle subtended at the centre by any given arc is twice any angle subtended at the circumference by the same arc (the key proposition), the related propositions (angles in same segment, in semicircle, in inscribed quadrilateral) may be treated as obvious corollaries. Time is saved and order introduced. The contrasted methods have been analogically described as 'a string of sausages and a bunch of bananas'.

The early teaching of algebra is deceptively easy. In the interests of clearness and definiteness I venture to stress and illustrate by reference to algebra the maxims that I have already formulated. Once again and with all the force at my command, I insist that there is no such thing as the right method. The approach here described is illustrative only, designed to add precision to general advice. The guiding aim is to secure the interest, confidence and understanding of the class; the outstanding danger, a mechanical handling of meaningless symbols.

First, the purpose of algebra is illustrated. It is not necessary at this stage to give a long talk about the place of algebra in the modern world, or an account of its development. Both of these will grow out of later experience. A simple example—'think of a number, double it, add 6' and so on—provides interest for the group, and enables smarter youngsters in the final lesson to make up similar problems for themselves. Practice with these for a few days will give the class confidence and a sense of achievement. Incidentally, they will see and use such mathematical symbolism as is contained in, say, $\frac{2n+6}{3}$, $3(n+4)$, without any preliminary fuss or to-do. Smarter pupils may try harder examples, or challenge the skill of the teacher with puzzles. The prestige of algebra is established.

Now, how does algebra do this? The point here is that algebra is a shorthand. Commercial classes learn to write shorthand, and then to transcribe (i.e. translate) it into ordinary writing. Musicians likewise use a special shorthand which has to be thoroughly learned. So we must learn the shorthand of algebra.

Thus:

(1) Ordinary English: two numbers added together, or the sum of two numbers. Shorthand: $a+b$, $x+y$, or any other two letters.

(2) Ordinary English: one number multiplied or divided by another. Shorthand: ab or $\frac{a}{b}$ (cf. $\frac{3}{4}$).

The important points here are that the class *uses* these abbreviations without long explanations, that the various possible combinations are all practised till they are done correctly, smartly, mechanically, which is not done in a day or a week, but requires the constant daily short practice to bring mastery—and that translation exercises are as important as the writing in symbols (e.g. explain in ordinary English, $3b + 4a$). Such exercises lead naturally to the use of formulae, which are likewise shorthand statements. Note here that the class already knows some formulae—e.g. those for areas of certain figures;[14] that other well-known formulae can now be introduced and examples worked out, thus giving a great sense of power and mastery, and that in some simple instances, the class can make up its own formulae—e.g. d(istance) = s(peed) × t(ime) i.e. $d = st$. Much practice with formulae—substituting values, altering the form, translating—is essential. Again, it is not all done as a separate and complete exercise.

These three types of exercise will be practised concurrently and steadily till they are handled with some confidence. They develop naturally in two directions:

(a) the transcription into shorthand of problems leading to simple equations. This power of transcription is of vital importance. The class learns early to handle it in both directions. There is no need at this stage actually to solve the problems, though this can sometimes be done readily as for example when a class of 'very weak' girls who evolved the equation $3x + 2 = 2x + 8$ completed the solution by arguing thus: If three times a number with 2 added to it is equal to something, it will be equal to 2 less than this if 2 is *not* added. So $3x = 2x + 6$. Similarly, dropping twice the number, $x = 6$. The 'mechanical operations' approach to algebra hides from children the significance of the symbolism, and leads to confusion.

(b) given, say, $x + y = 12$, the class, reading this as 'the sum of two numbers is 12', will readily provide the usual positive numbers. Suppose we now represent these values on squared paper. Form II work, especially in social studies, will probably have introduced pupils to the general idea of graphical representation. It is well to make sure, by preliminary practice, that the class can locate any given point ($x=3$, $y=9$). The straight-line graph (produced in both directions) introduces two new ideas—the existence of intermediate values ($x=2\frac{1}{2}$, $y=9\frac{1}{2}$) or functional change, and the concept of negative numbers.

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[14] It would possibly be preferable in primary schools to be satisfied with the *statement* (e.g. area of a rectangle = base × height), leaving the introduction of the formula proper to secondary school teachers.

But throughout, the elementary practice of the earlier processes must be constantly and regularly continued. Mastery comes with familiarity and practice. There is nothing in all this work to tax the reasoning powers of children. It is all a matter of practice and familiarity leading to confidence and skill.

I do not propose to offend the reader with further elaboration. There is, however, one suggestion that may be helpful in dealing with beginners. It is the old sporting maxim: Don't change a winning game, change a losing game. The meaning thereof—while the going is good, don't interrupt this work with introductory work on geometry and arithmetic, as set out in your time-table. Take this work daily till you get to a stage where you can leave it for a while to consolidate subconsciously. Take up your neglected geometry, and handle it similarly. An experienced teacher knows when the critical changing point has been reached. This advice is heterodox, but it is psychologically sound. Time-tables should be made for pupils, not vice versa.

The points that I would note in connection with the whole procedure are these: the class works from the very first lesson; constant revision leads to skill and confidence; the insistence on the meaning of symbols and on translation makes for understanding; later work is unobtrusively anticipated, so that new sections do not appear out of the blue; the grouping and linking of work gives the feeling of unity; no attempt is made to get 100 per cent accuracy in one type before the next is begun—overlap and repetition are the secrets;[15] formal and abstract statements are eschewed, accurate but homely language being preferred; mastery of skill is sought, not theoretical explanation; the handling of symbols is introduced without explicit rules; progress and success are stressed throughout.

The particular introduction which I have sketched is based on what has been actually accomplished by a very slow class of girls, who responded with enthusiasm. The absence of confusing rules for procedure was welcome and did not prevent success.

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[15] Chrystal, in the preface to Vol. 2 of his *Algebra*, quotes: '*Allez en avant, et la foi vous viendra*'.

V

A MISCELLANY

§1 TEACHER TRAINING

'TEACHERS OF MATHEMATICS NO LESS THAN TEACHERS OF ANY other subject in any type of school should possess certain qualifications which are not specific to the subject which they teach. They should be persons of character, of personality, and of experience. . . . Another qualification of a general kind which prospective mathematics teachers should possess is a broad cultural education.' In these words I. S. Turner opens his report on the training of mathematics teachers.[1]

In New Zealand, university graduates who aspire to become secondary school teachers are assessed for their general qualifications by district selection committees. Brief interviews with the applicants supplement consideration of their academic qualifications and of their supporting recommendations. The procedure, though necessarily hurried and not without weaknesses in the technique of interviewing, has proved fairly satisfactory in practice—mainly, perhaps, because of the proved quality of all the applicants. Candidates who have successfully specialized in mathematics at the university are, as a class, serious students of considerable intellectual powers. On my fifteen years' experience with graduates, I would venture the opinion that these students are a more evenly capable group than any other. Their commonest weakness is too narrow specialization. Men who sat additional mathematics for the Entrance Scholarship Examination and then proceeded to an Honours degree in mathematics—with perhaps physics and radio physics as subsidiary subjects—have obviously a limited range of teaching subjects and (probably) of interests. They are rather more tempted than others to fail in the human approach because of their highly abstract university training, their dominant interest in topics remote from the experience of school children, their familiarity with a type of symbolic thinking that in its rigour and objectivity is foreign to the average immature and untrained mind, and their relative weakness in powers of self-expression and of exposition. In extreme instances they stubbornly resist any suggestion of tempering abstract method to the frailty of youth. Their contact, during their year of professional training, with specialists in other subjects and with the more general problems of teaching aims and methods should be particularly beneficial. But the gain is not unilateral: they greatly strengthen the whole graduate group. Incidentally, they have regularly shown an interest in music, and even a degree of skill in musical performance that are markedly above the average of the graduate group.

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[1] National Council of Teachers of Mathematics, *Fourteenth Yearbook: The Training of Mathematics Teachers for Secondary Schools*, 1. New York 1939.

A statement by Professor Bullen[2] seems to me to be an excellent description of the ideal teacher of mathematics:

(i) He should be competent as a mathematician; at the very least he should be able to solve problems neatly and elegantly.

(ii) He should be something of a philosopher; in particular he should have some appreciation of the connection between

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[2] See Report of his address in the *Australian Mathematics Teacher*, November 1948.

mathematics and scientific inference in the broadest sense.
(iii) Not merely should he be a mathematician in the sense of having digested some ten or a dozen or so books on algebra, geometry, calculus, differential equations, etc.; he should have gone far enough to have some glimpses of an enthusiasm for living mathematics.
(iv) He should be able to impart his knowledge. For this purpose, he should be given the fruits of past experiences of mathematics and other teachers.
(v) He should be interested in educational research. Education as well as mathematics is a living thing and should not be treated as static.

The meaning and significance of the term 'living' as applied to mathematics was illustrated as follows: a teacher versed in relativity theory would view the teaching of parts of elementary analytical geometry in quite a different light from the usual. A body of teachers versed in relativity and the accompanying pure mathematics of tensors, would probably introduce suffix notation into school algebra at a fairly early stage. If Professor Bullen were a teacher of 13- or 14-year-olds, he felt that his reading in quantum mechanics would colour the way in which he dealt with certain questions involving measurement, and possibly add some life to his teaching, whatever damage he might do in other ways. Again acquaintance with the problems of mathematical philosophy would help a teacher to answer questions by intelligent boys in elementary school geometry; and so on.

Dr Turner[3] enunciates nine principles of training for the teacher of mathematics. I summarize these.
(1) Thorough previous training in mathematics.
(2) This training should be given by a university professor.
(3) Mathematics teachers should study the important

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[3] op. cit., 218–9.

branches of pure mathematics, mechanics, the history of mathematics, applications of mathematics, and especially the essential connection between the various branches of advanced mathematics and their counterparts at the more elementary stage.

(4) The mathematics teacher should make a less intensive study of some (preferably) closely related subject.

(5) The teacher during his active teaching service should strive to progress in his acquaintance with mathematics.

(6) A period of professional training is necessary.

(7) The content of a professional course should be organized principally to train teachers to teach mathematics.

(8) But mathematics teachers should also be equipped to teach a second (preferably allied) subject, and they should be trained professionally in this also.

(9) The training should include some courses in the theory and practice of education and in psychology.

Of these requirements, the first four are met in our university colleges, which give young mathematicians an admirable foundation on which to build. The most conspicuous omission is a systematic treatment of the history of mathematics, and this can readily be given in the year of professional training. The fifth requirement refers to later professional life, and teachers' training colleges can only encourage the ideal of a lifelong devotion to the study of mathematics, already engendered at the university. The remaining requirements, numbered (6) to (9), mark out clearly the task of the teachers' training college. A few comments on their significance are added.

It is conceded that college lecturers must keep always in touch with prevailing practice and with existing conditions, but it must be stressed that a college which merely inculcated approved methods would fail in its real purpose. Training colleges should be leaders in educational reform—'growing points in education'. Their lecturers should be soundly versed in educational principles and encouraged to keep in touch with changes in method. So, they will not equate either newness or age with virtue, but will be educationists in their own right. Such methods as they may favour will not be dogmatically imparted, or presented as the last word. As the Board of Education memorandum says, 'Uniformity in details of practice is not desirable, even if it were attainable.' The importance of quality in a college lecturer—academic, professional, and personal—becomes obvious. School practice and the observation of experienced teachers at work with their classes are likewise of the first importance for students in training. A close liaison between school and college is so obviously desirable as to require no elaboration. To secure it, however, is a difficult task. Senior teachers are kept very busy with their school duties; and the presence of students and the need to give them periods of supervised teaching practice are serious considerations. I seize this opportunity to pay a tribute to the unpaid service given in this way to our graduate students by teachers in our practising schools. Without their help, our course would be crippled. By precept and example they have inspired and strengthened the teaching of many students.

Perhaps the most difficult task for the training college is to convince the young honours graduate that university methods are unsuited to third-form pupils and that the rigorous treatment of mathematics at this level will defeat his aims. The fruits of another's experience are rarely acceptable to youth. The change of attitude from that of a student to that of a teacher is effected commonly with great difficulty.

Dr Bullen is reported as claiming that the student should review his mathematical knowledge with two purposes in mind.

First, he must make certain that he is prepared to answer in suitable language the good questions that his abler pupils will ask: In the term a^n, can a and n have any values? What is infinity? How many numbers are there? What are 0^0, etc.? What is i? Secondly, he must ensure that he has a firm grasp of the principles underlying the mathematical knowledge he has gained. To do this effectively requires a study of questions associated with the *Foundations of Mathematics*—the significance of the axiomatic development of mathematics, especially in its relationship to school geometry; the meaning of number and the development and extensions of the number concept; the meaning of proof; and the like. It also requires some study of the *History of Mathematics*, to gain knowledge and appreciation of the development of mathematical ideas, to put school mathematics in proper perspective (and how much of it is seventeenth century mathematics, or earlier!), and to provide interesting and illustrative 'asides' to enrich the teaching.[4]

In similar vein, I would express the opinion that the mere ability to work out most of the examples in Form II arithmetic is insufficient equipment for a primary school teacher. The belief that he needs no more background to his work than core mathematics—completed two years before he enters a teachers' college—seems to me to be educationally dangerous.

§2 TRANSFER OF TRAINING

Mathematicians have commonly claimed for the study of of their subject a specific virtue as a discipline in exact and systematic reasoning, and have rather assumed that the habits so formed overflow into the affairs of ordinary life.

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[4] loc. cit.

So they have justified the traditional study of algebra and geometry even by pupils who will almost certainly make no direct practical application of their learning to the business of their lives and indeed cannot claim to have made much progress in learning itself. In so doing, they have made two major assumptions—that the pupils do actually establish certain desirable habits and receive a discipline in mathematical reasoning, and that such an educational experience results in a general mental discipline that overflows into other forms of mental activity.

The first assumption has obviously not always been realized in classroom practice. Many pupils have failed to understand or appreciate mathematical reasoning and methods, and have largely wasted their efforts, even from the narrow criterion of progress in school mathematics. An outstanding teacher of mathematics, whose doubts about the disciplinary values of mathematics are quoted below, writes as follows: 'I think that today there is still a good deal of waste effort. In many lower grade classes the attempts at algebra (even suitably treated on "core" lines) are so pitiful that I think it would be wise to omit this subject altogether.' Such a realistic statement is encouraging. It admits failure when it occurs, and by implication stresses the difficulty of really successful teaching and the need for well-trained, efficient, versatile teachers. Failure is, of course, not confined to the teaching of algebra or of mathematics alone; but we need not worry about that. The question to be answered here deals with the second assumption: does the pupil who makes real progress in the study of mathematics at school receive a training that is valid and influential in other fields of activity? In the words of my correspondent: 'Just how much value is mathematics to a lawyer or a carpenter or a business man or a doctor? Is its only value the directly useful parts? I have heard many attempts at answering such questions, but never one definite authoritative reply. If the study of formal geometry is quite useless, I should like to know it as a fact.' This question puts into everyday non-technical language the problem known to the psychologist as the 'transfer of training'. Is there such a thing as a trained mind, irrespective of its content? Does a sound course in mathematics make the student merely (to that extent) a sound mathematician, or does it make him an abler thinker and worker in whatever he may be called upon to do? Do we carry over to the problems of life in general the methods and attitudes learned in specialized studies? I shall try to answer this clearly and simply in non-technical language.

The orthodox reply to this query, up till the end of last century, was an unqualified and unhesitating 'yes'. This belief was regarded as the fundamental justification of the grammar school type of education. Beginning in 1901, however, experimental work appeared to shatter completely any hope of the general transfer of training, and in America the results were used to justify the introduction of the unit or credit system, designed to prepare students for the specific occupations they intended to follow. Later and more significant experiment, however, has shown that these early findings were too sweeping. They applied to a limited and relatively unimportant aspect of education. The answer now becomes much more complicated. It is no longer an unqualified 'yes' or 'no', but resolves itself into an attempt to state specifically under what circumstances and to what extent transfer may occur. Hence arises the unsatisfactoriness of the answers as judged by a teacher seeking a simple and unqualified statement of 'fact'.

In general terms and put briefly, the answer of psychologists runs somewhat as follows. Transfer does not occur automatically or completely. The amount of transfer from one form of activity to another is greatest between activities having most in common. It takes place through the medium of the higher mental processes such as generalization and the recognition of principles running through various and varied applications; and it is greatly helped by the conscious recognition of, and emphasis on, the common elements or the similarities between the two situations involved.

To take a simple specific example. The study of Latin affords very considerable transfer to the later study of Greek, which has a very similar grammatical structure and some community of vocabulary and ideas. It transfers somewhat less readily to the study of French, where the grammatical structure of Latin has been considerably modified and a modern vocabulary and train of thought have developed. It will transfer less readily still to a Teutonic language such as German; less still to languages such as Chinese or Russian where—I imagine—the whole cast of thought is in a different mould. Yet all languages as such have certain elements in common: all attempt to express in words the ideas of those who speak them, all have some grammatical structure that enables them to indicate differences between, for example, present past and future, active and passive. The study of any one language therefore is in itself to some extent the study of all languages—some educationists today even advocate the generalized study of language before the study of any one foreign language. Finally, to continue the exposition, the common elements between the study of Latin and that of of, say, geography, mathematics, or music are much fewer, being limited to the experience and habit of systematic and sustained study, with some incidental contacts such as technical vocabularies.

It appears from this illustration that, as it is manifestly impossible for schools to give specific instruction in all possible subjects and to meet all conceivable situations, the main fields of human endeavour should be studied, as far as practicable, and conscious efforts should be made by the teacher of each subject to ensure the understanding of principles, the recognition of similarities in and contacts with other subjects, and the formation of sound methods of study and application. In this connection I would recommend to the mathematician the following two brief non-technical articles which appear to me to be sound, suggestive, and easily read:

Mathematics in the Secondary School, the report of a sub-committee set up by the Australian Council for Educational Research to study the teaching of mathematics in the secondary schools of Australia, together with an essay on 'Mathematics as a Training for Reasoning' by H. M. Campbell; and *The Content of Education*, Interim Report of the Council for Curriculum Reform, especially Chapter 8.

There are, however, very great difficulties in the way of carrying out the recommendations of these writers. I instance a few, without elaboration. The successful learning of mathematics does depend ultimately on the thorough mastery of certain specifically mathematical techniques and skills, and these must be thoroughly mastered as such. The proposed treatment throws a much heavier burden on the teacher who, as a specialist, is probably neither particularly interested in nor conversant with the associated subjects. Supplementary subject matter, as in the form of secondary school bulletins, would probably be necessary. Textbooks would require to be still further modified on the lines of the new education, examination aims and techniques would be profoundly influenced, and the whole syllabus of work would require revision. A particularly serious obstacle is to be met in the specialized needs of those who will carry their mathematical studies to an advanced stage. We cannot prejudice their interests, though practical experience might well show a gain rather than a loss from the new approach. Finally, the breaking down of boundaries between subjects, though ideally desirable, is in practice a dangerous and difficult procedure. What is everybody's concern is apt to be nobody's duty. The fundamental question is, perhaps, at what stage genuine specialization in treatment should begin. With sixth forms specialist treatment appears to be inevitable.

One further aspect of the question of transfer of training still needs consideration. Can the 'mind' be 'strengthened' by being subjected to systematic discipline? My correspondent puts this question also very clearly in the form of a specific example. He writes: 'What about geometrical riders? Every mathematics teacher I know thinks skill in these is one of the main things in geometry as a subject. Why is a boy a better person because he can solve riders? He is undoubtedly fairly intelligent, but has the study of geometry improved him? And if so, to what extent? I have never been able to find anyone to give me definite answers to these questions. Many educational books discuss the matter, but no one ever seems to decide it.'

The mathematician needs no formal approval from the psychologist for his faith in riders. Practice in solving these strengthens the grasp of the principle involved, ensures that the learner really understands and can apply his theorem, gives an active rather than a passive approach to the subject, thus making for successful learning of geometry—our first assumption. Mathematics is essentially a subject of skills and understanding: emphasis on power rather than on reproduction of memorized propositions is fundamental to success in learning.

But does such power 'improve' him generally? Is he any the better for his knowledge of geometry—apart from its practical applications? The hidden reference here may be to the innate intelligence of the child, or to his power to use it.

The simple explanation is that 'intelligence' as measured by the psychologist is a very different thing from what we commonly regard as intelligence. By definition, it excludes acquired knowledge or skill and indeed is further limited by the exclusion of special 'gifts' such as musical ability. The whole concept is based upon mathematical analysis of test results, and cannot be satisfactorily understood apart from a thorough study of factor analysis. The practical situation, however, can be described in popular language somewhat as follows.

Nature has set limits to the intellectual powers of us all. We cannot exceed those limits—cannot by exercise increase, as it were, our innate mental horsepower. But such power neither develops nor is exerted *in vacuo*. It can be, and is applied to the accumulation of knowledge and to the attainment of skills. The acquirement of these is a function of schooling. Analogically, one may note that the efficiency of a manual labourer may be increased not merely by strengthening his physical powers but by providing him with better tools, by teaching him sound methods of handling them, and by giving him an incentive to work.

The practice of solving riders teaches geometrical methods, and these are a powerful tool in the handling of mathematical problems. How far increased ability to handle mathematical concepts can be used in other activities raises again the old question of transfer which, as we have seen, is contingent and variable in amount.

§3 ACCURACY

With as much accuracy as can be expected from human frailty.[5] Accuracy has been claimed as the soul of mathematics, and most teachers of the subject deplore its absence from even the mechanical operations of secondary school entrants. There can be no doubt that many of these children are exasperatingly slow and uncertain in handling the simplest calculations, and that the new approach to arithmetic (particularly its lowly estate) makes it possible for teachers to accept too low a standard. Some, indeed, misunderstanding modern psychology, fear to insist on mechanical drill, the only firm foundation, and all are affected by the heavy and varied demands on the energies of children now made by the whole curriculum.

Third form teachers may, however, innocently overestimate the arithmetical weakness of their entrants. The long summer vacation tends to lower mechanical accuracy, where constant practice is so important. The changed circumstances—of school, teacher, and classroom method—tell further against good performance. A classification examination in the first week does little justice, therefore, to the primary schools, though for severely practical considerations it can hardly be avoided. In the solution of problems attention is necessarily directed from mechanical operations—no longer an end in themselves—and 'careless' errors normally appear. In the same way, children make spelling mistakes in essays or dictation that they would not make in a pure spelling test. A thorough mastery of the fundamental mechanical operations in arithmetic can be secured in the primary school, where repetition and routine are welcome as ends in themselves. With adolescents, this mechanical success is insufficient. George Hogben, Director of Education from 1899 to 1915, used to hold it against youth as its outstanding weakness that it was so tolerant of its own failings, so willing to accept error. A period of inaccuracy and unevenness of work is, in fact, typical of the adolescent—a normal and healthy, though irritating reflection of his own inner lack of confidence. It should disappear, along with his hobbledehoyism, as he matures.

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[5] *Memorandum on Algebra Teaching*, 37.

This desired mastery of the mechanical operations of arithmetic can be secured in the primary school without violating all the canons of the new education. It is achieved in some schools, though in how many I cannot state as we have now no universal measuring stick in the form of a leaving examination. Performance in third forms often reveals, however, a distressing lack of thoroughness and grasp. This may be partly due to a lack of co-operation between intermediate and secondary school, partly to poor primary work, partly to secondary school methods, and—perhaps more than we realize—to psychological factors. The misunderstandings that so often arise over questions of standards of work may resolve themselves finally into complaints about human nature. I do not know.

The span of attention is limited; hence the importance of a thorough mastery of the fundamental mechanics which must reach the stage of automatism. Haste likewise detracts from accuracy, and secondary teachers, familiar with the work of older pupils, may unwittingly push their new pupils too fast. This danger is intensified by the fact that the boys are beginning several new subjects to which they naturally pay special attention. If, as commonly happens, the atmosphere of the third-form classroom is more formal and restrained than that of Form II, a further handicap is introduced.

Secondary school teachers possibly overlook, too, the disturbing influence of unaccustomed written work. In primary schools, this occupies much less time than formerly. With those of us who live and work pen in hand the written page is the normal medium, and we may reason more accurately on paper than in speech. A certain kinaesthetic pattern in writing is developed with much practice, so that the hand appears to know the way, and manage itself. Without that practice, conscious control is necessary and this makes for slowness and error. Shorthand teachers handling speed practice know empirically what this means, and others may compare their skill in writing their signature with their slower and more uncertain efforts to write unfamiliar words with awkward letter-combinations such as *van Zyl, phthisis, ecthlipsis.* Perseveration tests depend on this fact.

Possibly the most successful teaching technique is to give constant practice and revision, to insist on form and method in setting out work and on neatness and clearness in symbols and figures, regular practice of 'mental', the use of estimates and checks as a matter of routine, constant reference to the realities behind symbols, and individual treatment associated with diagnostic tests. It is interesting to note that in the junior primary school arithmetic books—e.g. Standard 3—pupils are taught to check subtraction by addition, whereas in the Form II book checks are not mentioned. This appears to be an example of that lack of persistence in practising skills barely mastered and habits insecurely formed that leads to trouble in the post-primary schools. Buckingham[6] would extend the idea of drill to include other aspects of work: setting out of work (correct use of the symbol =), systematic checking (*when* and *how* to check, and practice in doing it). The *habit* of careful and methodical working, and of routine checking may well be established in the lower forms. The moral attitude towards accuracy develops more slowly. It is, perhaps, beyond the possibilities of average humanity. Certainly, it cannot be developed in the mathematics room alone, but it is a common task of all teachers.

§4 POSTULATIONAL THINKING

> *But if you think that Euclid represents absolute truth—then you are in a mess. Actually Euclid himself only said,* 'If *you admit certain things, then you must admit. . . .*'[7]

All our thought is ultimately postulational; but it is only rarely that we realize the nature and extent of our presuppositions. In everyday life we accept conventional beliefs and employ modes of thought that we never question. It is the province of the philosopher to probe into these particular assumptions, and few of us feel impelled to travel his difficult road or are capable of the abstractions that are his data. But in ordinary social life we must, in a humbler way, follow his method of thought. *If* we choose one profession or occupation, *then* we must accept a certain course of action; if we spend our money in one way, if we vote for a certain political party, if we accept office on a committee, if we deliberately insult our neighbour, if (in chess) we move our queen to . . . then. . . . Whenever we have to decide between different courses of action, this postulational attitude is forced upon us.

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[6] B. R. Buckingham, 'What becomes of drill?' National Council of Teachers of Mathematics. *Sixteenth Yearbook*, Ch. IX. New York 1941.

[7] W. W. Sawyer, *Mathematician's Delight*, 25.

As a generalized training for this important habit of thought, mathematical instruction may well become a most powerful weapon of the school teacher. In it, factual information is reduced to a minimum and attention is focused on the discovery of 'necessary and sufficient' conditions and on the drawing of conclusions. The use of precisely defined terms and of symbolism enables the crux of an argument to be very clearly indicated. This distinctive feature of the subject is indeed the source of its most serious limitations, as of its greatest virtues. The problem of the transfer of mathematical training to the handling of everyday affairs is intensified. The teacher should be vividly aware of this. Save for the exceptional few who will later find in mathematics their soul's final satisfaction, his pupils will shortly abandon the realm of systematic abstract reasoning. If their system of thought has been so tied to mathematics that it cannot be generalized the teacher's efforts have largely failed. It is precisely because this failure so often occurs that we must improve our approach to mathematics.

In geometry, we still have explicit reference to basic assumptions. The modern geometrician objects to Euclid's axioms and postulates, and discovers flaws in some of his arguments, but follows the same logical method. His assumptions are followed to their necessary conclusions. In algebra, (and later, of course, trigonometry) the pupil is likewise trained to use a reasoning tool of extreme subtlety. He applies it to 'given' problems, where he must accept the conditions of the problem, and follow them to their legitimate conclusion. Personal bias and social attitudes are irrelevant. He may hope that A is a faster runner than B, or that the area of the square is greater than that of the circle; but he must accept the data, and reason accordingly.

It is interesting to note here that a certain mental maturity is necessary before the 'given' will be accepted. The educationist Ballard tells somewhere of the mental tester who gave to a child the 'absurdity' test—'a woman was cut into several pieces by a train, and killed at once. Her body was taken to the hospital, and she was not expected to recover. What was absurd about that?' The youngster aptly replied, 'You are.' Renwick[8] cites these: 'John has 1/6. Mary has 4/-. What fraction is John's money of Mary's? . . . the children would probably say, "John's money is not a fraction of Mary's. It's his own." ' This aptly illustrates the ambiguity of the word *of*, 'the nasty word of mathematics'. Its unrelated uses (1) to mean 'multiplied by' and (2) merely to introduce a phrase, will always lead to confusion.

§5 COMMON SENSE IN MATHEMATICS

I can prove that the triangles are equal, but they aren't.[9]
I can do *these sums, but I can't understand them*![10]

No teacher needs convincing that the average pupil is capable of the most grotesque absurdities in mathematical calculations. Nothing is too impossible to be offered in an examination. University Entrance examiners have recently quoted the following:

Cost of a gas fire burning for a year found to be £18,474,968·75 or alternatively 13/6; the path round a field wider than the field itself; the handicap given in a race longer than the race; interest (actually below £20) found to be £51,249,999,999,999,853 5s.; and the length of a railway embankment (obviously less than 440 yards) found to be 3 ft. or 37,188 miles.

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[8] E. M. Renwick, *The Case Against Arithmetic*, 53. London 1935.
[9] *Teaching of Geometry*, Second Report.
[10] Renwick, op. cit., 1.

It would be ridiculous, of course, to make much out of such errors as these. Every subject produces in examinations a similar crop of howlers, without which examiners might well have their sanity impaired. But it would be unwise to accept them with resignation and even with complacency. The psychologist learns much about normal conduct from extreme cases, wherein normal weaknesses are exaggerated. Similarly, these mathematical atrocities have their significance for the classroom teacher.

We must allow for the unfavourable circumstances—the fear and excitement of a public examination on the result of which so much may depend, the form and ceremony of the new ordeal, the long and unaccustomed silence in the big (usually strange) hall, the printed papers, the presence of so many other candidates, the pronounced emphasis on time-limits, the frequently unexpected nature of the questions, the length of each paper, the supervision by a stranger, so precise and distant. One might almost claim that this alone explains the poor performance, and urge that more frequent and realistic preparation would remove the faults by removing the strangeness of the experience.

But it is not only in public or class examination that absurdities occur, and it is not only fearful excitement that produces them. Limiting our consideration to mathematics, we may note fear of the subject itself (the symbolism, formalism, strangeness of matter), ignorance of the topics implied (interest, discount, volumes, systems of measurement), weakness in fundamental operations (mechanical operations, conversion of decimals, use of logarithms, and the like), reliance on mechanical rules to obviate the need for thought (stereotyped problems, book-work memorized but not grasped), treatment of mathematics as an end in itself instead of a tool to solve real problems (lack of practical application, answers checked by key instead of by 'proving' or testing), natural inability (immaturity, lack of interest, poor general intelligence), and a general weakness in spatial and numerical concepts.

Only an experienced teacher can vividly realize the pitfalls and snares of the road, that so readily make of mathematics an unreal subject with arbitrary laws of its own. It is a pity that so many pupils get a false conception of the whole subject in the first years of school.

§6 APPROXIMATION

The use of approximation in junior arithmetic is beset with difficulties; but its importance overrides all objections to its use. I would note two types: the making of a preliminary rough result, and approximation proper—the recognition of the limits of accuracy.

The practice of making a rough preliminary estimate is commonly advocated by teachers, but usually with indifferent success. Pupils are apt to regard it as an imposition, a kind of double working of the same problem, a waste of time, and a meaningless procedure. Examiners have noted that where this approximation is required, candidates may offer an approximation and then provide a 'correct' answer that has no resemblance to the estimate. The discrepancy appears to raise no doubts.

The estimate may be of two kinds. It may be a rough guess based on knowledge of the situation. In the instances quoted above, for example, pupils should have some idea of common handicaps in a race, of the normal rates of interest, of the width of a path, and so be led to distrust ridiculous answers. Where, as in mechanical problems, any answer is apparently possible, some liberties must be taken with figures. Thus $91{\cdot}26 \div 7{\cdot}2$ is obviously somewhere about 13 (i.e. $91 \div 7$).

Approximation in the second sense of the term is a more ticklish concern and requires some understanding of the significance of numbers. Again there are two types: mechanical exercises, as where two decimals are to be multiplied or divided correct to so many decimal places, and problems where the data are themselves inexact. The first type is mechanical, and if the right method is followed is quite straightforward, though it must be admitted that any but the very simplest examples bristle with difficulty, e.g. a correspondent offers me the following: 'Using contracted multiplication and division, what degree of accuracy must be employed to evaluate $(27{\cdot}75881 \times 0{\cdot}6014523 \div 3{\cdot}4597)$ correct to four significant figures?' Even capable and experienced examiners can set a bad example—e.g. giving $\pi = \frac{22}{7}$ and then asking a result correct to four significant figures. As van Zyl points out, most teachers themselves do not understand the matter. The second raises the whole problem of *pure* and *applied* mathematics. The pupil must realize from the data whether the figures mean exactly what they say. Statistical figures (population, exports, imports, etc.) are obviously approximate. All actual measurements are likewise only approximate and calculations based on them are necessarily inexact.

It is an accomplishment to convince a pupil that the answer cannot be more accurate than the original measurement;

so that if the linear measures are correct only to the first decimal place, the area (calculated by their product) cannot be correct to more than one decimal place. Figures are figures to the average pupil, and any recognition of limits or error is regarded as in some way dishonest. Exercises in approximation (e.g. calculate cost to the nearest pound, penny, etc.) are not alone sufficient. The pupil is here still calculating exactly what he is told to do, and may still secretly regard the answer as strictly incorrect. Exercises in practical work, e.g. cost of papering a room, may be used to impress on pupils that the amount of material needed is more than a mathematical calculation of area will demand—owing to inevitable wastage, and that in buying material, commercial practice must be followed (buying in rolls of given widths and lengths, and with costs liable to slight modifications). A further useful illustration is the engineer's factor of safety.

This contrast between mechanical accuracy and the accuracy of reality may be distressing to pupils at a certain stage. It should not, however, be particularly so where commonsense methods are adopted. The Spens Report has an important note on this general topic. It says: 'In the early stages of arithmetic, the main aim is to secure exactness when working with exact numbers. In later arithmetic where the data are often only approximate, the aim should be to secure the utmost reliable information from approximate data. Textbooks often stress the inverse of this: given exact data, find an approximate answer. Many of the commercial rules in the books are opposed to modern commercial practice; if included in the general course, the commercial methods should be adopted.'[11]

[11] Spens Report, 241.

It is of interest to note that in the University Entrance Scholarship Examination, one mark (out of a maximum total of 2400) may separate a successful from an unsuccessful candidate.[12] Total scores in this examination are remarkably close. The 1949 results show the lowest two scholarship winners as scoring 1547 and 1546, the next (unsuccessful) candidate 1543. As a scholarship must either be given or not given, it may be that this cannot be avoided; but the unreliability of every examiner's marks, the inevitably different standards in different subjects, the different combinations of subjects taken, and the arbitrary maxima assigned to subjects all make the final placing a matter of considerable luck. The moderating of papers, the attempted standardizing of marking by the chief examiner and the final re-scaling of marks are palliatives: they do not reduce the margin of error to anything approaching an accuracy of 1 in 2000. The mechanical additions of scores are no doubt accurate; but what of the scores themselves? (An experienced New Zealand examiner comments: 'The conclusion as to the number of candidates who pass or fail in School Certificate by pure accident is shocking.')

§7 FEAR IN MATHEMATICS

Thou, drede, guo out myd guode wylle. . . . Vor
ryghtuolle loue deth out drede.[13]

The twin attitudes of fear and hatred are all too commonly engendered by the study of mathematics in a secondary school. For this unfortunate result, we have to blame several factors. First we have its forbidding symbolism. 'I like arithmetic, but I *loathe* π!' says Margaret.[14] When unfamiliar symbols replace words—themselves familiar symbols—difficulties begin and fear rather than interest or curiosity may develop. How many people are frightened away from music by its notation? How many are turned from the study of Greek or Russian or Chinese by the strange and formidable appearance of the alphabet that must first be mastered? How many youngsters hate chemical formulae? Little wonder that the mass of new symbolism in an algebra textbook puts the fear of death into timid souls. There are not even pictorial illustrations to give a feeling of warmth and comfort.

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[12] cf. J. H. Murdoch, *The High Schools of New Zealand*, 373.

[13] 'Thou, fear, go out with good will. . . . For righteous love casts out fear.' Dan Michel of Northgate, *Sermon on Matthew* xxiv, 43. A.D. 1340.

And this pardonable fear is strengthened by other causes. The traditional textbook is set out in cold, formal, condensed, purely intellectual and logical fashion. Its problems do not arise naturally, but are artificially prepared and hurled at the beginner. There is little that is human, and less that is humorous in the whole dreadful business. The formal incantations of geometry and the complicated rules of algebraic manipulation shake the beginner's confidence. It is so easy to be wrong and so hard to be right that he is never sure till the correct answer is given. Where the teacher insists on pedantic accuracy in form or expression, beyond the grasp of his pupils, his very zeal becomes fussiness and a handicap to the class. Even the anxious parent interesting himself in homework then finds his help unavailing; for, '*He* won't accept that—that's not the way we do it at school.' The proving of the apparently obvious, now fortunately dropped, added further to the burden of doubt and fear.

The insistence on a formalism and a logic beyond the comprehension of the beginner is still a temptation to teachers. Familiarity with the subject blinds them to the confusing nature of all symbolism. Blind and mechanical operation of rules and memorized reproduction of book-work are the sole hope of the bewildered pupil, who never knows when some new rule or unexpected complication will upset his whole work. A book such as Renwick's *The Case Against Arithmetic* is worth a night's reading, for it illustrates the confusion of mind and the baffled logic of children facing too soon the mysteries of symbolism.

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[14] Renwick, op. cit., 109.

Consider the following examples from this book:

$\frac{1}{5} \times 10 = 2$. 'You can *call* it multiplying, but it's really dividing!' (p. 58).

$3x + 2x = 5x$. 'What has become of the other x? The three and the two make five, but you had two x's, and now you have only one!' (p. 87).

$a - b$. 'It ought to mean "a multiplied by b", because when you put a letter beside another you have to multiply them.' (p. 89).

'If two minuses make a plus, two pluses ought to make a minus.' (p. 135).

Such examples illustrate the total obfuscation that so often resulted from the old approach to algebra. Mathematics is a language, and until we can translate to and from that language with ease and certainty, the attempt to use it as a tool for discovery is doomed to ignominious failure.

§8 HOW IT IS DONE

Charles Lamb in his essay, 'Imperfect Sympathies', has pictured the typical Scot (alas, my countrymen!) in words that might well apply to the traditional teacher of mathematics: 'His Minerva is born in panoply. You are never admitted to see his ideas in their growth—if indeed they do grow, and are not rather put together on principles of clockwork. You never catch his mind in an undress. . . . He does not find, but bring. You never witness his first apprehension of a thing. His understanding is always at its meridian—you never see the first dawn, the early streaks. . . .' This completeness is a serious pedagogic error. The magician draws his rabbit from the hat in sight of all; and till the trick begins to pall, we duly applaud. But how was it done? My mathematics professor, drawing his solution from the data, would blandly explain, 'It's just a dodge.' Precisely. But how did he come to think of the dodge? That is the pupil's problem.

It is obvious that success depends partly on native aptitude, partly on systematic practice. About the former, the teacher can do little or nothing beyond shaping his teaching practice accordingly. About the latter, he can do a very great deal indeed. In the first place, he must give frequent well-graded exercises. If from the first the pupil is encouraged to take even the simplest steps into the unknown on his own initiative, he is laying the foundation of mathematical skill. Clever children may be encouraged and intrigued by baffling problems; average children need the constant stimulus of success.

Let us take geometrical riders in illustration. They are very troublesome to children. If in the earliest lesson care is taken to avoid formality and abstractness, the feeling of strangeness and fear need not arise. Problems worked orally, without fuss and strain, and with acceptance of the obvious, induce the feeling of confidence and reality. Problems should be many, and simple; and the slower children must not be bustled. The writing of exercises should be delayed. It is helpful if the class draws for itself all the conclusions that it can, instead of facing prepared exercises.

When, later, more formal work is faced, the fruits of such practice will be gathered. There is no fear. What is left to the teacher now is to see that certain standard procedures are regularly followed. These include, of course, such procedures as recalling propositions or similar exercises that may help, drawing diagrams and marking in known facts, taking extreme cases, searching for 'necessary conditions', using indirect methods where the direct approach fails, assuming the result and seeing what it would involve.

Guidance along these lines is commonly given from time to time by teachers, but perhaps not often enough or emphatically enough to ensure that the methods become routine procedure. In particular, when a class has failed to solve a problem, the teacher is apt to give a solution or at least the vital hint. I suggest that he should do neither. He should frequently show the class how his mind works towards a solution even when this involves, as it usually will, the deliberate suppression of a known construction or solution. Let him think aloud before the whole class, demonstrating in action (not words) the rules he follows: first, the direct attack; this failing, the use of constructions; failing again, assumption of the conclusion and working back; failing again, some more accurate drawings till insight comes. Even complete temporary failure occasionally would do no harm. Similar class attempts on the blackboard, where individual suggestions are carried out, even though the master knows they will fail, with others trying their own solutions—a genuine co-operative effort—will help in the same manner. Not, 'What is the solution?' but, 'How shall we find a solution?' is the real problem.

Experience in coaching suggests that many weaker pupils are hopelessly handicapped in their efforts by unfamiliarity with the direct applications of propositions and important exercises. Frequently the incantation of a general enunciation obscures its real meaning in practice. Through specific applications, generalized statements come to assume real significance. In certain types of problem more detailed principles of guidance can be given. For example, where we are asked to show that a certain line passes through a certain point, the normal procedure is to draw a line through the point and prove that this is the line. Or, again, in a circle there is only one fixed point, the centre. In locus problems, therefore, the centre is apt to be connected to specified points or lines. In converse propositions, indirect proof by exhaustion may be necessary.

I suggest here that Polya's *How to Solve It* will help the teacher along the recommended lines. Polya's guiding rules, set out on the inside covers back and front, should be considered. Two of his problems that I have used successfully with students are: to inscribe a square in a given triangle, and to arrange the numbers 1–9 in any order and combinations (using each figure only once) to make a total of 100. Riddles are particularly useful in this respect; and if pupils can bring problems to the master, instead of vice versa, the resulting conference is all the more useful.

Impress on pupils that a methodical procedure must be followed, that the better we know our subject the greater our chances of success, and that when all else fails we must fall back on native wit and determination. Anything is better than the pose of omniscience by the teacher, and the suggestion that mathematics is a procession along a predetermined, mile-stoned and sign-posted road to the final comfort of Mathematicians' Inn, where struggles are no more.

Mechanical rules assume the garb of common sense when we understand what they mean. The modern approach, which begins with problems and formulae, is favourable to this understanding. Only let us make haste slowly in these early days. I suggest that in the early years the practice of translation into and from symbols should be a daily exercise. The mastery of new symbols cannot be acquired in a few weeks; and until symbols are an aid to working they are a hindrance and a danger. As Polya writes:[15] 'The intelligent student is justified in his aversion for algebra if he is not given ample opportunity to convince himself by his own experience that the language of mathematical symbols assists his mind.' You must be done with fear is the refrain of *Mathematician's Delight*; and Sawyer's attitude towards the subject is in keeping. What one fool can do another can, is the encouraging slogan of another writer, introducing the calculus. Polya takes the beginner behind the scenes, and shows him how the whole thing began, and how it was developed. To follow this lead is not to be superficial but to be sensible. When fear is driven out, love may enter.

§9 PUZZLES AND FUN

The fear that the teaching of mathematics may get out of touch with the realities of life may be overdone. It may be conceded that in an external examination paper there should be no catch problems, but in class work there is some room for puzzles and paradoxes. Many class teachers bear witness that these—used, of course, in strict moderation—can add much interest to the work, even with pupils who would be regarded as dull. It is noteworthy that the general public is interested in mathematical puzzles such as appear in some popular magazines. Similarly, the paradoxes in logic put life into argument. The poor barber, for instance, ('Of all the men in the village, I naturally don't shave any of those who shave themselves, but I do shave all those who don't shave themselves') how does he get on? If he shaves himself then he doesn't, and if he doesn't then he does! And how much discussion has arisen over the old conundrum beginning: Sisters and brothers have I none. . . !

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[15] op. cit., 129.

Several types of mathematical puzzles may be used. There is, first, the mystifying type of solution or mind-reading illustrated in our early lesson in algebra: Think of a number. . . . A well-known gambling game with four piles of matches is based on the binary number system, and baffles the uninitiated. Then there are the startling proofs, common in algebra, of such manifest contradictions as that any two unequal numbers are equal, or that all positive whole numbers are equal. These demonstrate vividly the fallacy of dividing by zero, and the need to be constantly on the watch against unwarily falling into this error. Smart pupils may even produce puzzles of their own, as for instance the interpretation of 0^0.

Finally, there are the straightforward puzzles of the type: The bottle and its cork cost 1/1, the bottle costing 1/- more than the cork, what is the price of the bottle? Problems on averages readily cause trouble, e.g. A man drove his car one mile to the top of a hill at the rate of fifteen miles an hour. How fast must he drive down a mile on the other side to average thirty miles per hour for the whole trip of two miles?[16]

Even teachers of mathematics should sometimes relax, and there is much virtue in the advice recently given by an inspector to my students: 'Whatever you do, avoid plod, plod; plod, plod.'

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[16] from E. P. Northrop, *Riddles in Mathematics*. London 1945.

§10 EXAMINATIONS IN MATHEMATICS

It must always be an important part of a teacher's duty to prepare his pupils to face the external examinations that they wish to pass. It is therefore pleasing to find examiners seeking through their questions to encourage sound teaching methods—a difficult task. However, mathematics is now an optional subject, so that only those pupils who have some aptitude for it may be expected to face the examination. Such pupils may well be required to show some insight and mastery of method, as well as a knowledge of formal work. When weaker pupils had to face the mathematics papers, the position was somewhat different. Then, the conscientious teacher up to University Entrance standard could very justly argue somewhat as follows. At this stage in their learning, it is unreasonable to expect of all pupils the insight and imagination needed to solve riders and involved problems. It should be sufficient that they have experienced the discipline involved in the mastery of formal work. Examination conditions are so disturbing that a weaker pupil cannot be expected to use his newly acquired skills with cool judgment, so that many candidates will make a far worse showing than is fair to them. Often the merest hint, permissible in a classroom but not in an examination, might turn complete failure into success. It is therefore wisdom to see that candidates know their bookwork thoroughly. So long as this covers (say) 60 per cent of the examination questions, weaker candidates may hope to obtain the needed 50 or 45, the due reward of industrious if not inspired virtue. It is likewise fair, and indeed incumbent on the teacher to watch jealously the examiners' questions, so that nothing unusual goes unchallenged, and that the due proportions of formal work and applied work are observed.

But this *argumentum ad misericordiam* holds no longer; it was never more than a specious plea for the right to cram. Teachers are now free to develop mathematical skill in their pupils, in confidence that the new examinations may be taken *en passant*. Good third-year pupils should reach a standard well above a bare pass in the School Certificate Examination, without any resort to cram. There is even some danger that pupils of very moderate ability might be attracted to sixth-form mathematics because of a good School Certificate mark.

Two other matters may be briefly noted—the unreliability of examination marking and the use of diagnostic and prognostic tests. The former is now well recognized by official examining bodies, and elaborate steps are taken to secure consistency in marking. The main difficulty is to ensure that reliability is accompanied also by validity: the relative values of mechanical accuracy, formal correctness, sound systematic reasoning, mathematical method, and insight are not readily assessed but are highly significant. These relative values will change, indeed, with different pupils—academic and non-academic for example; and they will change, too, with the same pupils at different levels. There is, therefore, no fixed ideal proportion, even in theory. The new emphasis is steadily on skill and power, as against formal reproduction.

Diagnostic tests are at present too little known by teachers. They are designed not merely to indicate general weakness but to discover the underlying causes and the particular forms of such weakness. These causes often involve emotional as well as intellectual and scholastic factors, and their removal calls for individual work. It is far too commonly reported in the primary school that a pupil 'lacks number sense' or is 'very weak in arithmetic'. Both are too often face-saving diagnoses that really mean 'badly taught'. Causes of failure may readily be found in premature introduction of abstract number concepts, in failure to secure mechanical accuracy in elementary mechanical processes either through lack of intelligent drill or irregular attendance by the child, in unskilled teaching methods or in confusion of methods following changes in teachers or schools, and in lack of interest by the pupil sometimes to the extent of being an unconscious defensive refusal to learn.

Standardized tests in the four fundamental rules applied to third-form entrants will reveal a wide range of attainment and often (with individual pupils) an unevenness of mastery that must be cured to prevent later failure. Similar tests of mechanical operations will discover weaknesses in such processes as adding fractions, handling decimals, calculating percentages, averages, and the like. Such diagnosis is, of course, of little value if it is not followed by the necessary corrective work. Finally, standardized problem-solving tests are available. These should obviously be compared with the results of intelligence tests, for problem-solving calls for more than mechanical skill and the application of rules to fixed types of problems. Some factors clearly involved are the ability to understand the problem, some previous experience in problem-solving (with teacher guidance) so that the pupil has some idea of possible lines of attack, a good method of setting out data, which helps to simplify the problem, confidence in one's ability springing from former success and good training, the ability to recognize the relationship of evidence, i.e. to deduce a third statement from a given statement, a knowledge of the subject matter, e.g. the meaning of the terms used (percentage, average, etc.), accuracy in mechanical operations, which is helped by neatness, method, and the use of rough checks, and finally keenness and a desire to succeed. We must expect to find considerable differences in ability, but good teaching is a great help towards success for any child.

Prognostic tests are obviously more doubtful and dangerous, for the future has a trick of upsetting the most careful calculation. Modern tests of natural mathematical ability do, however, add to the precision of the more general intelligence tests; and the claims of such an educationist as Earle might well be tested in some of our secondary schools.[17] Incidentally, the limiting conditions (especially the time element) of the application of standardized tests drives the compilers into certain forms and types. Practically all scholastic tests suffer from this snap-judgment, one-word-answer type of problem; and their value is reduced accordingly. They should be used in conjunction with others of the old type.

§11 GIRLS AND MATHEMATICS

Throughout this book both teacher and pupil have consistently been referred to as 'he'—a practice adopted solely in the interests of convenience of expression. To the best of my knowledge, there is no psychological evidence that girls cannot do mathematics as well as boys; but there is much evidence to the contrary. The subject may interest them less, or have less practical value for them, but it certainly concerns them; for them, as for boys, the core prescription in mathematics is compulsory.

[17] See F. M. Earle, *Reconstruction in the Secondary School*. London 1944.

University Entrance Scholarship results may appear at first glance to reveal a decided mathematical superiority of boys, but this is a specious conclusion. In the first place, such candidates (both boys and girls) must be of very high intelligence, and we might expect *a priori* that more boys than girls would reach the necessary level, if there is any validity in the finding of some psychologists that variability in intelligence is greater in boys than in girls. Social considerations further encourage superior performance by boys. Many girls' schools lose their best pupils from the lower sixth form: some leave in order to enter training college, to take up nursing or commercial work, or to live at home; some as a reaction against a competitive system in which many girls are seriously overworked; and some for economic reasons—even today, a career is commonly regarded as imperative for a boy but not for a girl. Women teachers have pointed out to me the following relevant facts also—that fewer girls than boys select mathematics for scholarship, tending rather to language study; that in academic forms girls give more time than boys to non-Scholarship subjects such as music and handwork; that home life makes more demands on girls than on boys; and that it is the policy of many girls' schools not to enter girls for scholarships, but to encourage them to be content with the less ambitious course for university bursaries.

It may also be reasonably noted as relevant that the girls who do take mathematics as a scholarship subject take it as one subject only, whereas the strongest boys frequently sit also in additional mathematics and in mechanics. The boy mathematicians have really specialized in the subject, whereas the girls are language or science specialists. Today, in the bigger boys' schools at any rate, the cleverest third-form entrants are gently encouraged to take science and mathematics options, a practice not followed in girls' schools.

As things stand at present, however, we must recognize that mathematics has a more limited appeal to girls than to boys. A small but increasing minority of girls will wish to study the subject at university level, and for them the secondary schools must make ample provision. For the non-university girl, the case is different. Under existing social conditions, she is much less likely than her brother to be called on to apply her knowledge of mathematics practically. It may be that in years to come girls will pursue the various applications of mathematics as freely and normally as boys do now. Certainly women have already entered professions and occupations that a generation ago were regarded as men's clear and peculiar domain. And as the education of girls begins to reflect this new social order, we (or more accurately our successors) may find that even the 'natural interests' of boys in mechanics and girls in domesticity are really 'conditioned' by early training and the pressure of accepted custom. Wars (and we have the cheerful prospect of bigger and brighter wars in the future) have encouraged the idea that women can do what men can, and the pressure of economic circumstances seems to reinforce this view. However that may be, there is an increasing demand from commerce, industry, and social services for girls with some mathematical background.

Finally, because I believe that some mathematical insight, as distinct from mechanical arithmetical skill adds spice to life and opens the door to many and varied fields of reading—in social problems, for example—I raise the question of methods of teaching in girls' post-primary schools. Are these, perhaps, ill-adapted to the temperament of girls? Is there, perhaps, too much stress on formal accuracy and neatness as compared with power and insight? Is there too little reference to things that interest girls, too much drive, too much pure intellect?

I cannot even begin to answer these questions. There is certainly a dearth of well-qualified women teachers of mathematics, and one cannot fairly expect from a teacher of general subjects a teaching of mathematics that is enlightened, interesting, and inspiring. Perhaps in the training of more specialist teachers there is hope.

VI

THE MEANING AND PLACE OF MATHEMATICS IN THE POST-PRIMARY SCHOOL CURRICULUM

One of the hardest tasks that an expert in any subject can undertake is to try to explain to the layman what his subject is, and why he makes such a fuss about it.[1]

A person to be really educated *should have been taught the importance of mathematics as an instrument of material conquests and of social organisation, and should be able to appreciate the value and significance of an ordered system of mathematical ideas.*[2]

AS INDICATED IN THE PREFACE, THIS LITTLE BOOK IS RATHER AN essay on the teaching of mathematics than a piece of psychological research. It seeks to help the ordinary teacher in his classroom. So, even in this concluding chapter, no attempt is made to analyse the nature of mathematical thinking itself. A quarter of a century ago, Spearman published evidence that mathematical ability is not one single special ability, that, for example, the union of arithmetic and geometry as 'mathematics' seems rather to be one of practical convenience.[3] Refinements in the method of factor analysis have made possible a considerable advance in thought since Spearman's day. Interested readers are referred to F. W. Mitchell's study[4] which both illustrates the modern methods of analysis and suggests at least four component elements in mathematical thinking. An interesting investigation by A. M. Blackwell even suggests some possible slight sex differences.[5]

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[1] Sawyer, *Mathematician's Delight*, 138.

[2] Spens Report, 236.

[3] C. Spearman, *The Abilities of Man*, 232. London 1927.

[4] F. W. Mitchell, *The Nature of Mathematical Thinking*. A.C.E.R. Series No. 53, Melbourne 1938.

The views here expressed, though (I hope) well founded, are essentially mine, and may well be challenged and disproved or discounted by mathematicians and non-mathematicians alike. Thus the neglect of mathematics in teachers' training colleges, being sanctioned by the Education Department and accepted by college principals and staffs, must obviously be defensible. To me, the defence is unconvincing and unsatisfactory. I acknowledge, however, very grave limitations in my own mathematical knowledge and skill and in my psychological competence. Moreover, in educational theory generally, there is still abundant room for violent differences of opinion as to aims and relative values, even among acknowledged educationists.

For the sake of clearness, a brief statement of the guiding ideas elaborated in this book introduces this final chapter. The following claims are made:

(1) The time allotted to the study of mathematics in both primary and post-primary schools is very decidedly less under the new syllabuses than it was under the old, but the subject is still part of the compulsory core. It need not be studied beyond this level, and in fact many entrants even to teachers' training colleges have not studied it further. The subject—even in the form of arithmetic—does not appear on the time-tables of teachers' training colleges.

(2) In spite of their many virtues, the textbooks at present used in primary schools have serious weaknesses in form,

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[5] A. M. Blackwell, 'A Comparative Investigation into the Factors Involved in Mathematical Ability of Boys and Girls', in *British Journal of Educational Psychology*, Vol. X, Pts. II and III, June and November 1940.

matter, and method; and changed conditions in postprimary schools suggest that a new textbook on core mathematics is needed, though the time for its compilation may not yet be ripe.

(3) There is much room for improvement in our teaching methods throughout the whole post-primary school. The real purpose and significance of the study of mathematics have too often been obscured by unenlightened teaching. Much of the fear, hatred, contempt, indifference or opposition commonly shown by the adult community to mathematics springs from this poor teaching, not from the nature of the subject itself.

(4) The limited mathematical skill and knowledge represented by the core prescription are of value to any boy or girl who really gains them. Teachers of core mathematics should regard their task as one of great importance, not despise it as a Sisyphean task or as too trifling to merit their best efforts. This prescription represents the only significant direct and systematic contact with the world of number and quantity that average pupils may receive. Core mathematics should be taught with enthusiasm and hope by teachers who understand the interests and outlook of ordinary non-academic pupils.

(5) As a practical corollary of all this we must train our future teachers of mathematics more efficiently. They must be imbued with an enthusiasm for living mathematics—coupled, of course, with a broad outlook that will enable them to work in harmony and co-operation with their teacher colleagues.

(6) Two strongly contrasting attitudes towards the teaching of school mathematics are observable among teachers. The more conservative of these exalt the virtues of the traditional

approach, and tend to regard the 'new education' with a mixture of anger, fear, and contempt. The opposing school—liberal, psychological, radical—represents a strong reaction from the formal and disciplinary view of education. It is here suggested that the new approach provides a real hope for the teaching of mathematics. This approach has, however, the dangers of its virtues—it throws great responsibility on the teacher and makes heavy demands of him, it must be wisely interpreted and constantly modified in the light of its real aims, and care must be taken to uphold among teachers a worthy conception of mathematics and among pupils a real comprehension of the subject.

(7) It would be silly to claim that the study of mathematics has some pre-eminent virtue justifying its forcible imposition on all pupils. Yet it has an important place in any well-organized scheme of post-primary school education.

(8) Finally, many of the virtues commonly attributed to mathematical training are contingent only. The subject is essentially one of insight rather than of information and requires the mastery of certain specific skills. If these latter are not acquired, if the learning has been largely rote learning or uncomprehending memorization, mathematics is not being learnt at all. Conscious efforts should also be made to encourage the transfer of mathematical training to other fields of action.

In a book such as this, devoted to the teaching of a single subject, there is an obvious temptation to overstress its educational significance. The very fact that mathematics alone is considered gives it a somewhat unhealthy prominence and a suggestion of unique importance. It remains for me now to consider very briefly the legitimate claims of mathematics to an honoured place in the post-primary school curriculum. Until very recently the supreme importance of mathematics was accepted; today its pretensions are hotly disputed.

The specialist in any subject is apt, in his zeal, to fall into several false assumptions that vitiate his arguments. Because he has himself received his greatest mental stimulus in one field, he finds it difficult to allow others to find their satisfaction elsewhere, or even to believe that there can be comparable satisfaction elsewhere. An old Highlander used to insist to me, a youngster at school, that Gaelic was the language of Heaven, and—apparently confident that I should ultimately meet the need for it—constituted himself an honorary tutor. Again, to the specialist, his subject is easy and interesting and he tends to underrate the difficulties facing the average beginner. Further, if he is a teacher he has a professional and economic interest in the welfare of his subject; and certainly he has the emotional bias associated with personal achievement. Finally, he is tempted to ascribe to even a school knowledge of his subject virtues that spring from many adult years of devotion to it. From the textbooks of various mathematicians, it appears that the following broad educational gains are to be expected, in addition to the more immediate improvement in mathematical skill and knowledge—love of truth and accuracy, the habit of precise and systematic reasoning, the ability to select from rough data the significant factors, to follow postulational thinking, to subsume specific rules under broad generalizations, and, more generally, to develop the strictly scientific attitude. The mere listing of these virtues—attained so hardly and incompletely by so few even of the most gifted of adult students—indicates a statement of ideals and aspirations rather than of practical realization. And the fact that almost identical virtues are claimed for the study of his particular subject by every other specialist teacher confirms this criticism. I propose, therefore, as a non-mathematician more or less free—though not wholly so—from the temptations noted above, to state what appear to me to be the more or less distinctive and significant features of post-primary school mathematics.

This statement is an attempt, not to claim for the study of mathematics a superiority over all other studies, but to emphasize that it has its peculiar virtues, not so easily realizable in other fields. It has likewise its peculiar limitations and dangers, that must be recognized. Nor is the affirmation that it provides a specific training in, say, numerical work, to be regarded as a denial that numerical work enters into other subjects. After all, every school subject has its own specific aim, and we should no more study geography to improve our mathematics than we should study mathematics to improve our English. Yet the study of geography does involve mathematical skills, and the study of mathematics does require accurate comprehension of English and conciseness of expression.

Post-primary school teaching of mathematics deals specifically with numerical and quantitative ideas. In primary and intermediate schools, a start has been made with mechanical accuracy and skill in the handling of numbers and of simple standard numerical problems such as calculating areas of walls or interest on a sum of money. This is the alphabet of mathematics. In the third form, the pupil begins to develop his skill. He gains increasing mastery, let us hope, of the mechanical work already done, but advances into new territory. In algebra, he is freed from the tyranny of specific numbers, and given a new power to deal with numerical problems, a new insight into the nature of numbers. In geometry, he learns to recognize and reason about shapes and forms that are of fundamental importance to mankind.

Some of the beauties typical of mathematical thought can actually be enjoyed even in the third form—a neat and tidy proof as contrasted with a clumsy and laboured one, a generalization that sums up with beautiful economy a multiplicity of instances, a graphical representation of functional changes enabling intermediate values to be read, the simplicity of the metric system. Abler pupils will even sense the fascination of numbers, of directed numbers, of zero and infinity, of irrational numbers, and the like.

The thrill of personal endeavour and of increasing mastery is particularly to be sought at this stage. The teacher of mathematics is most fortunately situated in this respect. His textbooks are full of graded examples, suited to the stage and the ability of his pupils. There is relatively little call for him to talk and expound—working while the class sits back and dreams. Properly used, these exercises suit themselves to the needs and powers of the individual pupils. Thus in one period a weak pupil may solve a single and very elementary rider for himself while his abler classmate does ten examples, progressively more difficult. The teacher aware of the dangers of driving and incessant talking, and of the need to let the pupils work, will find in mathematics a particularly favourable field for the exercise of his self-restraint. Such a remark should not be misinterpreted. It is temptingly easy to teach mathematics from the comfort of a cushioned chair while the pupil goes on with 'the next example in the book'. Such conduct is not even a rationalization of educational theory: it is merely a parody of teaching. The pupil's opportunity to work is also the teacher's opportunity to diagnose individual weaknesses and to help judiciously. A text book used and not abused encourages such personal guidance.

I do not claim that the results described in these paragraphs are always or even typically achieved. But that does not mean that they are idealized and incapable of achievement. Many teachers have realized them, even with classes of very moderate ability, and many more can do so if they will.

A second feature of mathematics is the very slight demand that it makes on experience of life. It is the perennial problem of the teacher of English to find literature suited to the age and experience of his pupils. The geographer discusses types of climate, modes of living, production of crops, and manufacture of goods that call for wide knowledge of life—book knowledge is a poor substitute. The historian speaks of historical movements, of political and economic problems, of clashes of ideologies, of reform bills and the Atlantic Charter. To keep their treatment real and to avoid false suggestion is the constant preoccupation of such teachers.

This freedom from experience that comes only with years is shared also by music, and it is significant that young children of genius can distinguish themselves in these two fields—music and mathematics. This has, of course, its concomitant danger, that the demands of everyday life may be ignored by those who devote themselves too narrowly to these studies. The point here at issue, however, is that mathematics deals essentially with material well within the experience of children. That teachers have managed to render the subject meaningless and obscure to so many pupils is a monument to pedagogic perversity.

It should be noted, further, that this relative independence of wide social or scientific experience does not mean that mathematics itself is remote from life. The complaint that mathematicians have been too ready to live in an ivory tower has already been made. The fundamental knowledge required is that of arithmetical operations together with (for algebra) the realization that specific figures or ideas can be represented in a kind of shorthand, and (for geometry) the the recognition of a few plane shapes. The power gained from this study can be applied to situations as they arise; and I have pointed out earlier the mistake of using unfamiliar material to provide problems. Through specific instruction, I can learn to use skilfully saw and hammer and chisel. As occasion arises, I can apply that skill. There is no need to familiarize myself beforehand with all possible situations.

A third feature of mathematics is that in its definiteness and precision it provides a training in exactness for the young. Here again, music has a special affinity with mathematics. Just as $9 \times 7 = 63$, neither more nor less, and as $a^2 - b^2 = (a+b)(a-b)$ and no other product, so the young musician must not read C for C sharp, or lengthen a crotchet to a minim. This demand for accuracy can never be relaxed, and indeed is strengthened as the pupil makes progress. Thus, he soon learns that 16 is not merely 4^2, but $(\pm 4)^2$. A child of twelve years can calculate the simple interest on a sum of money for a given time as accurately though not as quickly as Einstein himself, and he is required to do so. In a world where absolute accuracy is so rarely attainable and the nearest possible approach to it so generally desirable, concentrated study in such a subject as mathematics has an obvious even if strictly limited place.

This definiteness and precision of mathematical concepts and methods leads the pupil further to the recognition of the certainty and inevitableness of his conclusions, given his premises. He knows, or should know, without reference to his teacher, that his conclusion is correct and inescapable. This is a matter of no small moment to a young student. 'I can see that the angle-sum of a triangle is two right angles, but why must it be so?' was posed recently by a beginner to a student teacher. When this inevitableness is really grasped, the significance of mathematical and scientific investigation begins to appear. Incidentally, as the Mathematical Association's second report on the teaching of geometry stressed, this demonstration of mathematical necessity following an apparent uniformity in experience is a much more valuable and convincing form of proof than its converse, i.e. where a general truth is demonstrated and then illustrated.

At the same time, the efficient teacher will check the natural tendency of youth to over-confidence. The demand for strict and formal proof is a safeguard, but occasionally it may even be necessary to go farther and demonstrate the possibility of false certainty. I refer here, not to careless errors or assumptions, but to errors based on ignorance of more general principles, as, for example, the apparently sound demonstration that $1=2$, or that all triangles are isosceles. At a much later stage the confusing doctrine of relativity enters in and calls for another orientation of ideas.

The development of the concept of proof constitutes one of the most important aims in the teaching of mathematics. This is illustrated in the form of geometry examination questions, which run typically: Prove . . .; given . . ., prove . . . . It is one of the failures of our class teaching that so often we have crammed children with bookwork on the assumption that they cannot be trusted to reason for themselves. The practice is not confined to mathematics, but is particularly blameworthy there, where the whole significance of the study depends on understanding rather than knowledge. Fortunately mathematics is a subject that lends itself very readily to a checking of the reality of this insight. How far pupils really gain this understanding depends to a great extent on the way in which the subject is presented.

There is no need to elaborate this last statement here. It forms the main theme of the book. But it may be noted once more that few textbooks make sufficiently explicit the concept of necessary and sufficient conditions, a fundamental one in mathematics, with important general applications. Again, few teachers give sufficient guidance to children in the art of solving problems, or make sufficiently explicit the role of mathematics as an instrument of investigation and discovery, facing constantly new problems and evolving new techniques.

On the other hand, the study of mathematics has severe limitations. Obviously it cannot provide the literary, musical, historical, geographical, and artistic training and knowledge that are provided by the other core subjects in the curriculum. Its restriction to numerical and spatial fundaments stresses an important but single aspect of life, and its predominantly intellectual character narrows still more its appeal to children. Its initial artificiality in form and symbolism is confusing, and it is so systematically developed that early failure is particularly difficult to overcome. Frequent absences from class and those interruptions to normal routine that appear now to be inevitable in schools play havoc with progress. Lack of manipulative skill in mechanical operations is of itself sufficient to cause failure in many instances, the symbolism of algebra cannot be effectively assimilated without steady effort, and the formalism of geometry even in its modern presentation is trying for beginners. In the thorough treatment of these difficulties, the thrill of mathematics, the joy of discovery, the appeal of beauty may readily be lost. So much depends on the first teacher, and with weaker third forms the teacher is apt to be inexperienced or mathematically ill-equipped. The average pupil, as distinct from the academic, is more tempted to express himself, to talk, and to read hurriedly, rather than to think cautiously, to check carelessness in expression or calculation, to be methodical. These latter habits are indeed strictly unnatural, the product of training, and relatively few adults ever become proficient in them, apart from the narrow field of professional or occupational problems. So the study of mathematics presents a forbidding appearance to the beginner. That appearance is all too easily converted into reality. The study is necessarily exacting. Apparent progress is so slow as to be discouraging. The same textbook can be used for years, a term's work may be represented by a few pages, a period may be spent on one problem stated in a line or two. There is so little relaxation of strain, such as may be provided so readily in other subjects. The psychological effect on young minds is important. A sympathetic, careful, enlightened introduction is, in the circumstances, of great importance. Unfortunately, that is one of the rare experiences of timid youth. A psychologically sound approach to an essentially logical study is not easily made.

It will readily be conceded, too, that the more fundamental virtues claimed for the study of mathematics require for their realization some years of systematic study. Very weak pupils, who enter post-primary schools ill-grounded in mechanical arithmetic and who remain one or two impatient years, will at best master little more than some further skill in handling numbers. At the same time, the absence of any external examination enables those who will not proceed beyond the core prescription to be given a living introduction to mathematics which will really extend beyond mere manipulation of figures.

Again, there are many forms of reasoning in life that are barely represented if at all in school mathematics. This springs from the specific nature of mathematical treatment. School problems are necessarily simplified in form and substance to suit the needs and powers of beginners. A few modern textbooks are introducing examples where too little information is supplied and the pupil must discover what is missing, and others where unnecessary information is supplied. This, however, is a most unusual practice and from the standpoint of the teaching of mathematics has certain dangers. In life, on the other hand, we are baffled by a superfluity of unselected data that may or may not have any relevance to our problem, so that we are driven to guesswork, to weighing probabilities, or to mere opinion emotionally charged. Even worse, we must frequently weigh up conflicting factors that appear to have no common denominator, to balance against one another ethical, sentimental, rational, aesthetic, economic, and physical considerations. Here the other core subjects provide more direct contact with everyday problems and situations and together provide a wider and more general appeal. Finally, the practice of mathematics calls for the high development of certain technical skills that enter comparatively little into the problems of ordinary life. Where pupils do not get beyond these skills, there is legitimate complaint that their study has been of little value.

One last danger in mathematical study should be noted: it may lead to an over-exaltation of mathematical procedures and a contempt for subjects of study not amenable to its methods. Some of its leading exponents, to judge by their published opinions, are not guiltless in this respect.

In this essay, I have tried to keep my feet on the ground, to avoid overstatement and yet to do justice to the place of mathematics in the post-primary school curriculum. I have urged its value for the average pupil as well as for the more gifted, and have stressed the need of more enlightened teaching methods. Pure mathematicians, I fear, will have found me an unworthy champion, and even my colleagues on the language side may regard me with some suspicion. Let me finish with a reference to the joys of pure mathematics as set out by a notable exponent.

Professor G. H. Hardy offers a delightful defence of his life.[6] He underlines the claim that pure mathematics is of no practical value ('Judged by all practical standards, the value of my mathematical life is nil'), that its 'tremendous effects' (Whitehead's phrase) 'have been, not on men generally, but on men like Whitehead himself'. His type of mathematics he designates 'real', as opposed to the 'trivial' mathematics of Hogben and his school. The latter, covering 'useful' mathematics (he includes here 'the bulk of school mathematics', calculus, elementary mechanics and 'a fair proportion of university mathematics') is of little interest to him, as having 'no particular aesthetic merit'. 'A mathematician, like a painter or a poet, is a maker of patterns' which must be both 'beautiful' and 'serious'—i.e. possess 'generality and depth'. His claim to recognition is essentially that of the great artist in any walk of life.

[6] G. H. Hardy, *A Mathematician's Apology*. Cambridge 1948.

I would end with this emphasis on the beauty and artistry of mathematics. For Hardy and his fellows, this implies philosophical mathematics. But even for us common mortals—yes, even for the dullards in Form III there is beauty in a generalization, however small, in a neat and final proof of the simplest mathematical proposition. Hardy himself offers the 'intelligent reader, however slender his mathematical equipment', Euclid's proof of the existence of an infinity of prime numbers and Pythagoras's proof of the irrationality of $\sqrt{2}$ as examples of pure beauty. Anyone with a feeling for mathematics must recognize the charm of these simple demonstrations as the intelligent listener or observer, however unskilled technically, knows when he is in the presence of a musical or artistic masterpiece.

It is my final plea that even the weakest of our students should be given the opportunity to see and admire some of the minor works of mathematical art, especially the simplest. The youngster who gets even the slightest thrill of discovery or mastery, be it from the most elementary of all mathematics, has lived through a significant experience.

APPENDIX

SOME SUGGESTIONS ON METHOD IN ARITHMETIC FOR FORMS II AND III

SOME AGREEMENT ON METHOD IS DESIRABLE IN THE INTEREST of children, and should not be regarded as an imposition on the teacher, or a check to his originality. Life would be safer and sweeter in Queen Street, so that the charge of glumness might even be repelled, if we were a little less individualistic in our habits, and condescended to observe traffic regulations occasionally. True originality, as distinct from mere freakishness, is built on observance of fundamental routine. Regional agreements on certain mechanical processes, at least between individual secondary schools and their contributing primary schools, is highly desirable. Three broad fields are distinguishable: the methodical setting-out of work, the actual method of working to be favoured, and the satisfaction of mathematical precision.

The first of these deals with such things as neat figuring and spacing, the correct placing of units, tens, etc., in addition, the use of working columns, the mathematical statement of problems, the naming of units employed, the keeping of work systematically together, the correct use of abbreviations, and so on. Agreement here should be secured without difficulty. From one such local agreement, I abstract the following important note on mathematical statement. 'This is the essence of correct setting out of all mathematics work. Any preliminary progress made by the pupil in this direction is of the greatest value. By the time Form II is reached, there are many types of arithmetical work that lend themselves to this type of setting out. . . . A mathematical statement requires the proper use of the sign = which is to be read only as, "is equal to". . . . Whatever is being dealt with, or is to be found, is the proper subject. This is always written first in word form. Thus the pupil must select the proper subject and write this first. Then follow any qualifying phrases, the verb =, and finally the proper subject in number or number and unit form. Thus an ordinary statement "Tea costs 2/- a lb." would be written: "Cost of 1lb. of tea = 2/-". Here proper subject in word form is "cost", the qualifying phrase "of 1lb. of tea", the verb is =, the 2/- is the proper subject in number and unit form. . . . Except for examples of mechanical processes, little is met by the pupil in arithmetic that cannot be handled in this manner, and once this plan of setting out is assiduously employed it is surprising how quickly a child adapts his thinking to this approach. . . .'

The actual method of working to be favoured will vary in individual instances in different regions. The use of the complementary method of subtraction is commonly advocated. The method of multiplication of decimals shown on page 35 of the Form II Arithmetic text and on page 42 of Kenyon's *New Outlook Mathematics*, Book I, may be rejected. The relative virtues of the two methods are discussed in the Mathematical Association's report,[7] and in Westaway's *Craftsmanship in the Teaching of Elementary Mathematics*.[8] The use of arrowheads in division of decimals is advocated by Godfrey and Siddons in *Teaching of Elementary*

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[7] *The Teaching of Arithmetic in Schools*, 27-9.

[8] pp. 82-4.

Mathematics.[9] The most acceptable method of decimalization of money, the use of factors in multiplication and division, and many other procedures, can well be decided by local groups — preferably after careful consideration of the authorities already quoted. In the simplification of fractions, it is advisable to avoid the term 'cancel' which obscures the basic principle and is a constant source of trouble later in mathematics. Durell in his *General Arithmetic* says: 'If unchecked, the pupil will apply the misnomer *cancel* to a variety of completely different mathematical processes, and furthermore commit a multitude of mathematical atrocities in its name, such as $\frac{\cancel{a}}{\cancel{a}+b} = \frac{1}{b}$.' Such topics cannot be pursued farther here. They, and many more of a like nature, are well discussed in the texts referred to.

The satisfaction of mathematical precision may at times appear a superfluity of virtue. But good habits can be formed almost as easily as bad, and accuracy is a fundamental quality of mathematical thought. Godfrey and Siddons write: 'It is quite true that many of the explanations that the arithmetic teacher makes will be apparently forgotten. Nevertheless they will stay in the pupil's subconscious mind; when algebra fractions are being done, the good teaching of vulgar fractions will bear fruit.'[10]

It seems a pity that primary school teachers of arithmetic and non-specialist secondary teachers should not be familiar with these standardized procedures, and that so many should, through ignorance or perversity, ignore details which mathematicians recognize as of the first importance. Sometimes a little submissive faith is of more value than self-assertion.

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[9] pp. 111–3. [10] op. cit., 16.

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I am particularly indebted to Mr R. P. Kania for the help he has given me in compiling the bibliography.

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